JOURNAL OF PURE MATHEMATICS

Volume 21, 2004

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA

35, Ballygunge Circular Road, Kolkata - 700 019, India

JOURNAL OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA

6

The Journal of Pure Mathematics publishes original research work and/or expository articles of high quality in any branch of Pure Mathematics. Papers (3 copies) for publication/Books for review/Journal in exchange should be sent to :

The Chief Managing Editor
Journal of Pure Mathematics
Department of Pure Mathematics
University of Calcutta
35, Ballygunge Circular Road
Kolkata 700 019, India

12/6 CUR-H00412-6-G1146091

Jour. Pure Math., Vol. 21, 2004, pp. 1-12

ON AN EXTENDED MULTIPLE HARDY-HILBERT'S INTEGRAL INEQUALITY

YANG BICHENG AND LOKENATH DEBNATH

ABSTRACT : This paper deals with an extended multiple Hardy-Hilbert's integral inequality with some parameters and a best constant factor involving The Gamma function. Several new particular results of the paper represent an extension as well as an improvement of earlier results.

Key words and pharases : Hardy – Hilbert's inequality, β function, Γ function
2001 Mathematics Subject Classification Codes : Primary 26D15.

1. INTRODUCTION

If $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n, b_n \geq 0$, $0<\sum_{n=1}^{\infty} a_n^p < \infty$, and $0<\sum_{n=1}^{\infty} b_n^q < \infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.1}$$

where the constant $\pi/\sin(\pi/p)$ is the best possible. Inequality (1.1) is well known as Hardy–Hilbert's inequality. Its integral form is as follows :

If $f(t), g(t) \geq 0$, $0<\int_0^{\infty} f^p(t)dt$, and $0<\int_0^{\infty} g^q(t) < \infty$ then

$$\int_0^{\infty}\int_0^{\infty} \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^{\infty} f^p(t)\, dt\right)^{1/p} \left(\int_0^{\infty} g^q(t)\, dt\right)^{1/q} \tag{1.2}$$

here the constant $\pi/\sin(\pi/p)$ is still the best possible (See Hardy et al. [1]).

The Hardy – Hilbert's inequality is important in analysis and applications (Mitrinovic et al. [2]). In recent years, Inequality (1.1) had been strengthened by Yang and Gao [3, 4]. Yang [5, 6] gave the following two distinct generalization of (1.2) :

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y-2a)^\lambda} dxdy < k_\lambda(p) \left\{\int_a^\infty (t-a)^{1-\lambda} f^p(t)dt\right\}^{1/p} \left\{\int_a^\infty (t-a)^{1-\lambda} g^q(t)dt\right\}^{1/q}; \quad (1.3)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dxdy < \frac{\pi}{\lambda \sin\left(\frac{\pi}{p}\right)} \left\{\int_0^\infty t^{(p-1)(1-\lambda)} f^p(t)\, dt\right\}^{1/p} \left\{\int_0^\infty t^{(q-1)(1-\lambda)} g^q(t)dt\right\}^{1/q} \quad (1.4)$$

where the constant factors $k_\lambda(p) = B\left(1-\frac{2-\lambda}{p}, 1-\frac{2-\lambda}{q}\right)$ ($\lambda > 2-\min\{p, q\}$) in (1.3) and $\pi / \lambda \sin(\pi / p)(\lambda > 0)$ in (1.4) are all the best possible and $B(u, v)$ is the Beta function. When $a = 0$, and $\lambda = 1$, both inequalities (1.3) and (1.4) reduce to (1.2).

Using the Gamma function, Hong [7, See (3.1) for $t = \lambda$] gave an extension of (1.2) in the form of multiple integral and proved the following integral inequality.

If $a \in R$, $a > 0$, $p_i > 1$, $\sum_{i=1}^n \frac{1}{p_i} = 1$, $r_i = \frac{1}{p_i} \prod_{j=1}^n p_j > b \geq 0$, $\lambda > \frac{1}{a}\left(n-1-\frac{b}{r_i}\right)$,

and $f_i \geq 0$ ($i = 1, 2, \ldots, n$), then

$$\int_a^\infty \int_a^\infty \cdots \int_a^\infty \left[\sum_{i=1}^n (x_i - \alpha)^a\right]^{-\lambda} \prod_{i=1}^n f_i(x_i)\, dx_1 dx_2 \ldots dx_n$$

$$\leq \frac{\Gamma^{n-2}}{a^{n-1}\Gamma(\lambda)} \frac{1}{a} \prod_{i=1}^n \left\{\Gamma\left(\frac{1}{\lambda}\left(1-\frac{b}{r_i}\right)\right) \Gamma\left(\lambda - \frac{1}{a} - (n-1)\frac{b}{r_i}\right) \int_a^\infty (t-\alpha)^{n-1-a\lambda} f_i^{p_i}(t)\, dt\right\}^{1/p_i} \quad (1.5)$$

We find that (1.5) neither reduces to (1.3) for $n = 2$, and $a = 1$, nor reduces to (1.4) for $n = 2$, $\alpha = 0$, and $\lambda = 1$ (in this case for $a = \lambda$), but reduces to (1.2) for $n = 2$, $\alpha = 0$, and $a = b = \lambda = 1$. It follows that inequality (1.5) is only an extension of (1.2).

In spite of Hong's work, some new extensions, improvement and new generalization seem to be important and useful for applications. So the main objective of this paper is to prove a new multiple Hardy-Hilbert's integral inequality with some parameters and a best constant factor so that we can make an extension of (1.3) and (1.4) and an improvement of inequality (1.5).

In order to prove main results, we need the following lemmas :

2. SOME LEMMAS

Lemma 2.1. If $k \in N$, $r_i > 0$ $(i = 1, 2, \ldots k+1)$, and $\sum_{i=1}^{k+1} r_i = \tilde{\lambda}(k)$, then we have

$$\int_0^\infty \cdots \int_0^\infty (1 = +\sum_{i=1}^{k} u_i)^{-\lambda(\tilde{k})} \prod_{i=1}^{k} u_i^{r_i - 1}\, du_1 \ldots du_\lambda = \{\Gamma(\lambda(k)\}^{-1} \prod_{i=1}^{\tilde{k}+1} \Gamma(r_i) \tag{2.1}$$

Proof. We may prove equality (2.1) by induction.

For $k = 1$, since $\tilde{\lambda}(1) = r_1 + r_2$, and (see Wang and Guo [8]),

$$B(p, q) = \int_0^\infty \frac{t^{p-1}}{(1+t)^{p+q}}\, dt = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} \quad (p, q > 0), \tag{2.2}$$

we have

$$\int_a^\infty (1+u_1)^{-\tilde{\lambda}(1)}\, u_i^{r_i - 1}\, du_1 = B(r_1, r_2) = \frac{1}{\Gamma(\tilde{\lambda}(1))} \Gamma(r_1)\Gamma(r_2).$$

Hence (2.1) is valid for $k = 1$.

If for $k \geq 1$, (2.1) is valid, then for $k+1$, since $\sum_{i=1}^{k+2} r_i = \tilde{\lambda}(k+1)$ setting

$v = u_1 / (1 + \sum_{i=1}^{k+1} u_i)$ in the following, we have

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \{(1+\sum_{i=1}^{k+1} u_i)^{-\lambda(\tilde{k}+1)}\}^{-1} \prod_{i=1}^{k+1} u_i^{r_i - 1}\, du_1\, du_2 \ldots du_{k+1} \tag{2.3}$$

$$= [\int_0^\infty \int_0^\infty \cdots \int_a^\infty \frac{1}{(1+\sum_{i=2}^{k+1} u_i)^{\tilde{\lambda}(k+1) - r_1}} \prod_{i=2}^{k+1} u_i^{r_i - 1}\, du_2 \ldots du_{k+1}]\, [\int_0^\infty \frac{1}{(1+\alpha)^{\lambda(k+1)}}\, \alpha^{r_i - 1}$$

In view of $\sum_{i=2}^{k+2} r_i + r_1 = \tilde{\lambda}(k+1)$, by (2.2), we find

$$\int_0^\infty (1+\alpha)^{-\lambda(\tilde{r}+1)}\, v^{r_1 - 1}\, dv = \frac{1}{\Gamma(\tilde{\lambda}(k+1))} \Gamma(\sum_{i=2}^{k+2} r_i)\, \Gamma(r_1). \tag{2.4}$$

Since $\widetilde{\lambda}(k+1) - r_1 = \sum_{i=2}^{k+1} r_i$, then by the assumption of induction on k, we find

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty (1+\sum_{i=2}^{r+1} u_i)^{-\lambda(k+\widetilde{1}) r_l} \prod_{i=1}^{k+1} u_i^{r_i-1}\, du_1\, du_2 \ldots du_{k+1} = \{\Gamma(\sum_{i=2}^{k+2} r_i)\}^{-1} \prod_{i=1}^{k+2} \Gamma(r_i) \qquad (2.5)$$

In view of (2.3), (2.4) and (2.5), we obtain

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty \frac{1}{(1+\sum_{i=1}^{k+1} u_i)^{\lambda(\widetilde{k}+1)}} \prod_{i=1}^{k+1} u_i^{r_i-1}\, du_1\, du_2 \ldots du_{k+1} = \frac{1}{\Gamma(\lambda(\widetilde{k}+1))} \prod_{i=1}^{k+2} \Gamma(r_i),$$

It follows that for any $k \in N$, (2.1) is valid by induction.

The lemma is proved.

Lemma 2.2. If $\alpha \in R$, $n \geq 2$, $a > 0$, $p_i > 1$ $(i = 1, 2, \ldots, n)$, $\lambda > -\min\limits_{1 \leq i \leq n} \{p_i\}$, and $\sum_{i=1}^{n} \frac{1}{p_i} = 1$, for $j \in \{1, 2, \ldots, n\}$, setting

$$\overline{\omega}_j = (x_j - \alpha)^{a + a(\lambda - n)/p_j} \int_\alpha^\infty \int_\alpha^\infty \int_\alpha^\infty \left\{ \frac{1}{\sum_{i=1}^{n} (x_i - \alpha)^a]^\lambda} \right.$$

$$\left. \times \prod_{\substack{i=1 \\ i \neq j}}^{n} (x_i - \alpha)^{a-1+a(\lambda - n/p_i}\right\} \quad dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n, \qquad (2.6)$$

then we have

$$\varpi_j = \frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(1 - \frac{n-\lambda}{Pi}\right) \qquad (j = 1, 2, \ldots, n).$$

Proof. Setting $\widetilde{P}n = Pj$, and

$$u_i = (x_i - \alpha)^a / (x_j - \alpha)^a, \; \widetilde{P}i = pi, \text{ for } i = 1, 2, \ldots, j-1;$$

$$u_i = (x_{i+1} - \alpha)^a / (x_j - \alpha)^a, \; \widetilde{P}i = P_{j+1}, \text{ for } i = j, j+1, \ldots, n-1$$

in (2.6) by simplification, we obtain

$$\varpi_j = \frac{1}{a^{n-1}}\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty}\frac{1}{\left(1+\sum_{i=1}^{n-1}u_i\right)^{\lambda}}\prod_{i=1}^{n-1}u_i^{(\lambda-n)/\tilde{P}_i}\quad du_1du_2\ldots\ du_{n-1} \tag{2.8}$$

Since $n \geq 2$, $\tilde{P}_i > 1$, and $\sum_{i=1}^{n}\frac{1}{\tilde{P}i} = 1$ then we have $\min_{1 \leq i \leq n}\{\tilde{P}i\} \leq n$,

$\lambda > n - \min_{1 \leq i \leq n}\{\tilde{P}i\} \geq 0$, $1 - (n - \lambda)/\tilde{P}i > 0$ $i = 1, 2 \ldots, n)$ and $\sum_{i=1}^{n}\left[1-(n-\lambda)/\tilde{P}i\right] = \lambda$,

substitution $n - 1$ for k, λ for $\tilde{\lambda}(k)$, and $1 - (n - \lambda/\tilde{P}i$ for r_i $i =$ 1, 2, ..., n) in (2 . 1), by (2 . 8), we get

$$\varpi_j = \frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-2}{\tilde{P}i}\right) = \frac{1}{a^{n-1\Gamma(\lambda)}}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-2}{Pi}\right)\ (j = 1, 2, \ldots, n).$$

Hence equality (2 . 7) is valid.

The lemma is proved.

Lemma 2.3. *If* $\alpha \in R$, $n \geq 2$, $a > 0$, $Pi > 1$ $\lambda > n - \min_{1 \leq i \leq n}\{Pi\}$, $\sum_{i=1}^{r}\frac{1}{Pi} = 1$, and

$0 < \varepsilon < \gamma = \left[\lambda - n + \min_{1 \leq i \leq n}\{Pi\}\right]/2$, then we have

$$\varepsilon\int_{\alpha+}^{\infty}\cdots\int_{\alpha+1}^{\infty}\frac{1}{\left[\sum_{i=1}^{n}(x_i-\alpha)^a\right]^{\lambda}}\prod_{i=1}^{n}(x_i-\alpha)^{a-1+a(\lambda-n-\varepsilon/a)/Pi}\quad dx_1\ \ldots\ dx_n$$

$$\geq \frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-\lambda}{Pi}\right) + o(1) + \varepsilon O(1)\ \ (\varepsilon \to 0^{+}) \tag{2.9}$$

Proof. By, (2.7) and (2.8), we have

$$\int_0^\infty \cdots \int_0^\infty \frac{1}{\left(1+\sum_{i=1}^{n-1} u_i\right)^\lambda} \prod_{i=1}^{n-1} u_i^{(\lambda-n-\varepsilon/a)/Pi} \, du_1 \ldots du_{n-1}$$

$$= a^{n-1}\varpi_n + o(1) = \frac{1}{\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(1-\frac{n-\lambda}{Pi}\right) + o(1) \; (\varepsilon \to 0^+). \tag{2.10}$$

And we find

$$A = \int_{\alpha+1}^\infty (x_n-\alpha)^{-1-\varepsilon} \left[\int_0^{1/(x_n-\alpha)^a} \cdots \int_0^{1/(x_n-\alpha)^a} \frac{1}{\left(1+\sum_{i=1}^{n-1} u_i\right)^\lambda} \times \prod_{i=1}^{n-1} u_i^{(\lambda-n-\varepsilon/a)/Pi} du_1 \ldots du_{n-1} \right] dx_n$$

$$< \int_{\alpha+1}^\infty (x_n-\alpha)^{-1-\varepsilon} \left[\int_0^{1/(x_n-\alpha)^a} \cdots \int_0^{1/(x_n-\alpha)^n} \prod_{i=1}^{n-1} u_i^{(\lambda-n-\varepsilon/a)/P_i} du_1 \ldots du_{n-1} \right] dx_n$$

$$\le \int_{\alpha+1}^\infty (x_n-\alpha)^{-1} \prod_{i=1}^{n-1} \int_0^{1/(x_n-a)^a} u_i^{(\lambda-n-\gamma/a)/P_i} du_i] \, dx_n$$

$$= \prod_{i=1}^{n-1} \frac{1}{1+(\lambda-n-\gamma/a)P_i} \cdot \frac{1}{a[n-1+(\lambda-n-\gamma/a)(1-1/p_n)]} = O_1. \tag{2.11}$$

Setting $u_i = (x_i - \alpha)^a / (x_n - \alpha)^a$ $(i = 1, 2, \ldots, n-1)$ in the left-hand side of (2.9), in view of (2.10) and (2.11), we obtain

$$\varepsilon \int_{\alpha+1}^\infty \cdots \int_{\alpha+1}^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^a\right]^\lambda} \prod_{i=1}^n (x_i = \alpha)^{a-1+a(\lambda-n-\varepsilon/a)/P_i} \, dx_1 \ldots dx_n$$

$$= \frac{\varepsilon}{a^{n-1}} \int_{\alpha+1}^\infty (x_n-\alpha)^{-1-\varepsilon} [\int_{1/(x_n-\alpha)^a}^\infty \cdots \int_{1/(x_n-\alpha)^a}^\infty \frac{1}{\left(1+\sum_{i=1}^{n-1} u_i\right)^\lambda}$$

$$\times \prod_{i=1}^{n-1} u_i^{(\lambda-n-\varepsilon/a)/Pi} du_1 \ldots du_{n-1}] \; dx_n$$

$$= \frac{\varepsilon}{a^{n-1}} \{ \int_{\alpha+1}^{\infty} (x_n - \alpha)^{-1-\varepsilon} [\int_0^{\infty} \cdots \int_0^{\infty} \frac{1}{\left(1 + \sum_{i=1}^{n-1} u_i\right)^{\lambda}} \prod_{i=1}^{n-1} u_i^{(\lambda - n - \varepsilon/a)/Pi} du_1 \ldots du_{n-1}] dx_n - A \}$$

$$\geq \frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(1 - \frac{n-\lambda}{Pi}\right) + o(1) - \frac{\varepsilon}{a^{n-1}} O_1\left(\varepsilon \to 0^+\right)$$

Hence putting $O(1) = -\frac{1}{a^{n-1}} O_1$, Inequality (2.9) is valid.

The lemma is proved.

3. MAIN RESULTS

Theorem. 3.1 If $\alpha \in R$, $n \geq 2$, $a > 0$, $P_i > 1$, $\lambda > n - \min_{1 \leq i \leq n} \{P_i\}$, $\sum_{i=1}^{n} \frac{1}{P_i} = 1$, and $f_i \geq 0$, such that $0 < \int_{\alpha}^{\infty} (t - \alpha)^{a(n-\lambda)-1-(a-1)Pi} f_i^{P_i}(t) dt < \infty$ ($i = 1, 2, \ldots, n$), then

$$\int_{\alpha}^{\infty} \cdots \int_{\alpha}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^a\right]^{\lambda}} \prod_{i=1}^{n} f_i(x_i) dx_1 \ldots dx_n$$

$$< \frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(1 - \frac{n-\lambda}{pi}\right) \left\{ \int_{\alpha}^{\infty} (t - \alpha)^{a(n-\lambda)-1-(a-1)pi} f_i^{pi}(t) dt \right\}^{1/pi} \tag{3.1}$$

where the constant factor

$$\frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^{n} \Gamma\left(1 - \frac{n-\lambda}{pi}\right)$$

is the best possible. In particular,

(i) for $Pi =$ n ($i = 1, 2, \ldots, n$), we obtain

$$\int_{\alpha}^{\infty} \cdots \int_{\alpha}^{\infty} \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^a\right]^{\lambda}} \prod_{i=1}^{n} f_i(x_i) dx_i \ldots dx_n$$

$$< \frac{\Gamma^n\left(\frac{\lambda}{n}\right)}{a^{n-1}\Gamma(\lambda)}\left\{\prod_{i=1}^{n}\int_\alpha^\infty (t-a)^{n-a\lambda-1} f_i^n(t)dt\right\}^{\frac{1}{n}}; \tag{3.2}$$

(ii) for $\lambda = n - 1$, we find

$$\int_a^\infty \cdots \int_a^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^a\right]^{n-1}} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n$$

$$< \frac{1}{a^{n-1}(n-2)!}\prod_{i=1}^n \Gamma\left(1-\frac{1}{Pi}\right)\left\{\int_\alpha^\infty (t-\alpha)^{(a-1)(1-pi)} f_i^{pi}(t)dt\right\}^{1/pi};$$

(iii) for $\lambda = n$, we have

$$\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^a\right]^{n}} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n$$

$$< \frac{1}{a^{n-1(n-1)!}}\prod_{i=1}^n \left\{\int_\alpha^\infty (t-\alpha)^{-1-(a-1)pi} f_i^{pi}(t)dt\right\}^{1/pi} \tag{3.4}$$

(iv) for $a = 1$, we have

$$\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left(\sum_{i=1}^n x_i - n\alpha\right)^{\lambda}} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n$$

$$< \frac{1}{\Gamma(\lambda)}\prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right)\left\{\int_\alpha^\infty (t-\alpha)^{n-\lambda-1} f_i^{pi}(t)dt\right\}^{1/pi}, \tag{3.5}$$

where the constant factors the above inequalities are all the best possible.

Proof. By Holder's inequalities, we have

$$\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^\alpha\right]^{\lambda}} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n$$

$$= \int_\alpha^\infty \cdots \int_\alpha^\infty \prod_{j=1}^{n} \{ \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^\alpha\right]^{\lambda/p_j}} [(x_j - \alpha)^{\alpha(n-\lambda)(1-1/p_j)+(\alpha-1)(1-p_j)} \times$$

$$\prod_{\substack{i=1 \\ i\neq j}}^{n} (x_i - \alpha)^{a-1+(\lambda-n)/p_i}]^{1/p_j} f_j(x_j)\} dx_1 \ldots dx_n$$

$$\leq \prod_{j=1}^{n} \{ \int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^\alpha\right]^{\lambda}} [(x_i - \alpha)^{\alpha(n-\lambda)(1-1/p_i)+(a-1)(1-p_i)} \times$$

$$\prod_{\substack{i=1 \\ i\neq j}}^{n} (x_i - \alpha)^{a-1+a(\lambda-n)/p_i} f_j^{p_j}(x_j) dx_1 \ldots dx_n \}^{1/p_j} \tag{3.6}$$

If (3.6) takes the form of equality, then there exist constants $C_j > 0$ ($j = 1, 2, \ldots, n$), such that (see Kuang [9]), for any $j, k \in \{1, 2, \ldots, n\} (j \neq k)$,

$$C_j \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^\alpha\right]^{\lambda}} (x_j - \alpha)^{a(n-\lambda)(1-1/p_j)+(a-1)(1-p_j)} \prod_{\substack{i=1 \\ i\neq j}}^{n} (x_i - \alpha)^{a-1+a(\lambda-n)/p_i} f_j^{p_j}(x_j)$$

$$= \quad C_k \frac{1}{\left[\sum_{i=1}^{n} (x_i - \alpha)^\alpha\right]^{\lambda}} (x_j - \alpha)^{(n-\lambda)(1-1/p_k)+(a-1)(1-p_k)} \prod_{\substack{i=1 \\ i\neq k}}^{n} (x_i - \alpha)^{a-1+1(\lambda-n)/p_i} f_k^{p_k}(x_k),$$

a.e. in $[\alpha, \infty) \times [\alpha, \infty) \times \ldots \times [\alpha, \infty)$.
It follows that

$(x_j - \alpha)^{a(n-\lambda)-(a-1)p_j} f_j^{p_j}(x_j) = F(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$,
a.e. in $[\alpha, \infty) \times [\alpha, \infty) \times \ldots \times [\alpha, \infty)$.

and $(x_j - \alpha)^{a(n-\lambda)-(a-1)p_j} f_{pj}^{p_j}(x_j) = F(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n), =$ constant,
a.e. in $[\alpha, \infty) \times [\alpha, \infty) \times \ldots \times [\alpha, \infty)$.

which contradicts the fact that $0 > \int_\alpha^\infty (t-\alpha)^{a(n-\lambda)-1\,1-(a-1)p_j} f_j^{p_j}(t) dt < \infty$.

Hence, by (3.6) and (2.6), we obtain $\int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^a\right]^\lambda} \prod_{i=1}^n f_i(x_i)dx_1 \ldots dx_n$

$$< \prod_{j=1}^n \{\int_\alpha^\infty \varpi_j \cdot (t-\alpha)^{a(n-\lambda)-1-(a-1)p_j} f_j^{p_j}(t)dt\}^{1/p_j}. \quad (3.7)$$

By (2.7), we obtain (3.1).

For $0 < \varepsilon < \gamma = [\lambda - n + \min_{1\le i\le n}\{p_i\}]/2$, setting $\tilde{f}_{i(x_i)} = (x_i-\alpha)^{a(n-\lambda)-1-(\lambda-n-\varepsilon/\alpha)p_i}$ *for* $x_i \ge \alpha+1$; $\tilde{f}_{i(x_i)} = 0$, *for* $\alpha \le x_i < \alpha+1$ $(i = 1, 2, \ldots, n)$, then we obtain the following equality

$$\varepsilon \prod_{i=1}^n \left\{\int_\alpha^\infty (t-\alpha)^{\alpha(n-\lambda)-1-(\alpha-1)p_i} f_i^{p_i}(t)dt\right\}^{1/p_i} = 1 \quad (3.8)$$

If there exist $\alpha \in R$, $a > 1$, and $\lambda > n - \min_{1\le i\le n}\{p_i\}$, such that the constant factor

$$\frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right)$$

in (3.1) is not he best possible, then there exists a constant K

$\left(0 < K < \frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right)\right)$, such that (3.1) is valid when we replace

$\frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right)$ by K.

It follows from (2.9) and (3.8) that

$$\frac{1}{a^{n-1}\Gamma(\lambda)} \prod_{i=1}^n \Gamma\left(1-\frac{n-\lambda}{p_i}\right) + o(1) + \varepsilon O(1)$$

$$\le \varepsilon \int_\alpha^\infty \cdots \int_\alpha^\infty \frac{1}{\left[\sum_{i=1}^n (x_i-\alpha)^a\right]^\lambda} \prod_{i=1}^n \tilde{f}_i(x_i)dx_1 \ldots dx_n$$

$$< \varepsilon K \prod_{i=1}^n \left\{\int_\alpha^\infty (t-\alpha)^{a(n-\lambda)-1-(a-1)p_i}(t)dt\right\}^{1/p_i} = K(\varepsilon \to 0^+).$$

Hence, we have $\frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-\lambda}{p_i}\right) \le K.$

which contradicts the fact that $K < \frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-\lambda}{p_i}\right)$

It follows that the constant factor $\frac{1}{a^{n-1}\Gamma(\lambda)}\prod_{i=1}^{n}\Gamma\left(1-\frac{n-\lambda}{p_i}\right)$ in (3.1) is the best possible.

The theorem is proved.

When $n = 2$ in Theorem 3.1, we obtain.

Corollary 3.1 If $\alpha \in R$, $p > 1$, $\frac{1}{p}+\frac{1}{p} = 1$, $a > 0$, $\lambda > 2 - \min\{p, q\}$, and $f, g \ge 0$, such that $0 < \int_{\alpha}^{\infty}(t-\alpha)^{a(2-\lambda)-1-(a-1)p} f^p(t)dt < \infty$, and $0 < \int_{\alpha}^{\infty}(t-\alpha)^{a(2-\lambda)-1-(a-1)q} g^q(t)dt < \infty$, then

$$\int_{\alpha}^{\infty}\int_{\alpha}^{\infty}\frac{f(x)g(y)}{\left[(x-\alpha)^{\alpha}+(y-\alpha)^{\alpha}\right]^{\lambda}}\,dxdy < \frac{1}{a\Gamma(\lambda)}\Gamma\left(1-\frac{2-\lambda}{q}\right)$$

$$\times\left\{\int_{\alpha}^{\infty}(t-\alpha)^{a(2-\lambda)-1-(\alpha-1)p} f^p(t)dt\right\}^{1/p}\left\{\int_{\alpha}^{\infty}(t-\alpha)^{a(2-\lambda)-1-(\alpha-1)q} g^q(t)dt\right\}^{1/q}, \tag{3.9}$$

where the constant factor $\frac{1}{\alpha\Gamma(\lambda)}\Gamma\left(1-\frac{2-\lambda}{p}\right)\Gamma\left(1-\frac{2-\lambda}{q}\right)$ is the best possible.

Remarks. (i) It is obvious that inequality (3.9) is an extension of (1.3) and (1.4), so is (3.1). (ii) Since the constant factor in (3.1) is the best possible, then (3.1) is a more accurate estimate than (1.5) (iii) Inequalities (3.2), (3.3), (3.4), (3.5) and (3.9) represents new results.

REFERENCES

1. G. H. Hardy J. E. Littlewood and G. Polya, *Inequalities,* Cambridge Univ. Press, 1952. MR 13 : 727c.
2. D. S. Mitrinovic, J. F. Pecaric and A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, Boston, 1991.
3. Yang Bicheng and Gao Mingzhe, On a best value of Hardy-Hilbert's inequality, *Advances in Math,* **26,** No. 2(1997).

4. Gao Mingzhe and Yang Bicheng, On tghe extended Hilbert's inequality, *Proc. Amer.* Math. Soc., **126**, No. 3(1998).

5. Yang Bicheng, On Hardy-Hilbert's integral inequality, *J. Math. Anal. Appl.*, 261(2001).

6. Yang Bicheng, On an extension of Hardy-Hilbert's inequality, *Chinese Anna. math.*, **22(A)**, No. 2(2002, to appear).

7. Hong Yong, All-sided generalization about Hardy-Hilbert integral inequalities, Acta *Math Sinica*, 44, No. 4(2001).

8. Wang Zhuxi and Gup Dunren, *An Introduction to Special Functions*, Science Press, Beijin, China, 1979.

9. Kuang Jichang, Applied Inequalities, Human Education Press, Changsha, 1993.

Department of Mathematics
Guandong Education College
Guangzhou, Guangdong 510303
People's Republic of China
E-Mail: bcyang@public.guangzhou.gd.cn

Department of Mathematics
University of Texax – Pan American
Edinburg, TX 78539-2999, U.S.A.
E-mail: debnath@panam.edu

Jour. Pure Math., Vol. 21, 2004, pp. 13-24

A HARDY-HILBERT'S TYPE INEQUALITY RELATED TO FIBONACCI SEQUENCE

GAO MINGZHE AND LOKENATH DEBNATH

ABSTRACT : In this paper, it is shown that Hardy-Hilbert's type inequality related to Fibonacci sequence can be established by introducing a parameter $\lambda \left(1 - \frac{q}{p} < \lambda \leq 2\right)$ and two functions $u(x)$ and $v(x)$ which are connected with Fibonacci sequence. In particular, for case $p = 2$, a new Hilbert's type inequality for double series related to Fibonacci sequence is built.

2000 Mathematics Subject Classification : $26D_{15}$ $40C_{15}$ $33B_{15}$

Key words : Hardy-Hilbert's inequality, Fibonacci sequence, double series, weight function, beta function.

1. INTRODUCTION

It is well known that the sequence of the form

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ \ldots \tag{1.1}$$

is called Fibonacci sequence. Its general term can be written in form

$$u(n) = \frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{1}\right)^{n+1} - (-1)^{n+1}\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\right\} \tag{1.2}$$

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers, $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$.

If $\sum\limits_{n=1}^{\infty} a_n^p < +\infty$ and $\sum\limits_{n=1}^{\infty} b_n^p < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq (\pi \csc \pi / p)\left(\sum_{n=1}^{\infty} a_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty} b_n^q\right)^{1/q}, \tag{1.3}$$

where the coefficients $\pi \csc \pi / p$ contained in (1.3) is best possible (sec [1]). It is the famous Hardy-Hilbert theorems for double series. In particular, for case $p = 2$, the classical Hilbert's double series theorem is obtained.

In general, Fibonacci sequence (1.1) doesn't seem to be connected with the inequality (1.3).

However the results appeared recently in some papers (such as [3]-[8] show that it is justifiable for us to establish a new Hardy-Hilbert's type inequality related to Fibonacci sequence.

For convenience, we define two functions by

$$u(x) = \frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^{x+1} - (-1)^{[x+1/2]+1}\left(\frac{\sqrt{5}-1}{2}\right)^{x+1}\right\} \tag{1.4}$$

and $$v(x) = \frac{1}{\sqrt{5}}\left\{\left(\frac{\sqrt{5}+1}{2}\right)^{x+1} + (-1)^{[x+1/2]+1}\left(\frac{\sqrt{5}-1}{2}\right)^{x+1}\right\} \tag{1.5}$$

When $x = n$, the function $u(x)$ become the relation (1.2) and $v(n)$ is one related to Fibonacci function (see (2.1)). Let $\alpha_r = 1 - \frac{2-\lambda}{r}$. Then $B(\lambda - \alpha_r, \alpha_r)$ expresses the beta function. In particular, when $r = p$, $B(\lambda - \alpha_p, \alpha_p)$ is denoted by B^*, when $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$ and $1 - \frac{q}{p} < \lambda \leq 2$. At same time we stipulate also that the sequences $\{a_n\}$ and $\{b_n\}$ are nonnegative. Throughout this paper, we will frequently use these notations and functions.

It is mentionable that Yang and Debnath [3] established new inequalities by introducing parameters A, B and λ :

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{(Am + Bn)^{\lambda}} < \frac{B(\alpha_p, \alpha_q)}{A^{\alpha_p} B^{\alpha_q}} \left\{\sum_{n=1}^{\infty} n^{1-\lambda} a_n^p\right\}^{1/p} \left\{\sum_{n=1}^{\infty} n^{1-\lambda} b_n^q\right\}^{1/q} \tag{1.6}$$

where the constant factor $\frac{B(\alpha_p, \alpha_q)}{A^{\alpha_p} B^{\alpha_q}}$ is best possible.

Besides, the following result was given in the paper [4] :

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\lambda}+n^{\lambda}}<\frac{\pi}{\lambda\sin\pi/p}\left\{\sum_{n=1}^{\infty}n^{(p-1)(1-\lambda)}a_n^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}n^{(q-1)(1-\pi)}b_n^q\right\}^{1/q} \tag{1.7}$$

where the constant factor $\pi/(\lambda\sin\pi/p)$ is best possible.

Their focus is to change the denominator of the function of the left-hand side of (1.3). Such as the denominator $(m + n)$ is replaced by $(Am + Bn)^{\lambda}$ in paper [3]; and the denominator $(m + n)$ is replaced by $m^{\lambda} + n^{\lambda}$ (λ is a parameter which is independent of m and n) in the paper [4] etc., such that some new results were yielded. Hence when the denominator of the function of the left-hand side of (1.3) is replaced by the Fibonacii function, a new inequality established is significant in theory and applications. The main purpose of the present paper is to establish the following inequality of the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}}\le k\left(\sum_{m=1}^{\infty}\omega_p(m)a_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\omega_q(n)b_n^q\right)^{1/q}, \tag{1.8}$$

where $u(m)$ and $v(n)$ are respectively functions defined by (1.4) and (1.5), and then to decide the coefficient k and to prove k to be best possible, at same time to find the expression of the weight function $\omega_r(x)$, ($r = p$, $x = m$; $r = q$, $x = n$). As application, we give a new Hardy-Littlewood's type inequality related to Fibonacci function.

2. LEMMAS

In order to prove our assertion, we need the following lemmas.

Lemma 2.1. *Let $u(x)$ and $v(x)$ be function defined respectively by (1.4) and (1.5). Then for any $n \in N$, the functions $u(x)$ and $v(x)$ are derivable at point $x = n$, and*

$$u'(n)=\left(\ln\frac{\sqrt{5}+1}{2}\right)v(n),\ v'(n)=\left(\ln\frac{\sqrt{5}+1}{2}\right)u(n). \tag{2.1}$$

It is very easy to prove this lemma. Hence it is omitted.

Lemma 2.2. *Let $r > 1$, $0 \le rs < 1$ and $\lambda > 1 - rs$. Then*

$$\int_0^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{rs}dt=B(\lambda-(1-rs),1-rs) \tag{2.2}$$

where $B(m, n)$ is the beta function.

Proof, According to the definition of the beta function we have

$$B(m, n) = \int_0^1 u^{m-1}(1-u)^{n-1} du.$$

Put $t = 1 / u - 1$, then

$$\int_0^\infty \frac{1}{(1+t)^\lambda}\left(\frac{1}{r}\right)^{rs} dt = \int_0^1 u^{\lambda-2+rs}(1-u)^{-rs} du.$$

This shows that the equality (2.2) is true.

Lemma 2.3. *let* $0 \le ps < 1$ and $1 - qs < \lambda \le 2$. Define a function Φ by

$$\Phi(s) = \{B(\lambda - (1 - ps), 1 - ps)\}^{1/p} \{B(\lambda - (1 - qs), 1 - qs\}^{1/q} \tag{2.3}$$

where $B(m, n)$ is beta function. Then $\Phi(s)$ attains the minimum B^*, when $S = \dfrac{2-\lambda}{pq}$.

Proof. Basing on the relation $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ where $\Gamma(z)$ is the gamma function, we can write (2.3) as

$$\Phi(s) = \frac{1}{\Gamma(\lambda)}\left(I_p^{1/p}\ I_q^{1/q}\right),$$

where $I_r = \Gamma(1 - rs)\,\Gamma(\lambda - (1 - rs))$, $r = p, q$

Taking the derivative of $\Phi(s)$ we have

$$\Phi'(s) = \Phi(s)\ \Psi(s)$$

where $\Psi(s) = -\Psi(1 - ps) + \Psi(\lambda-(1 - ps)) - \Psi(1 - qs) + \Psi(\lambda-(1 - qs))$, here $\Psi(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ is the psi function. We choose thus s such that $1 - ps = \lambda - (1 - qs)$, so that $1 - qs = \lambda - (1 - ps)$, hence $S = \dfrac{2-\lambda}{p+q}$. Since that $\dfrac{1}{p} + \dfrac{1}{q} = 1$, it follows that $S = \dfrac{2-\lambda}{pq}$.

We therefore have $\Psi\left(\dfrac{2-\lambda}{pq}\right) = 0$. i.e. $\Phi'\left(\dfrac{2-\lambda}{pq}\right) = 0$. It is known from the paper [9] that

$\Psi'(z) = \zeta(2, z) = \displaystyle\sum_{n=1}^{\infty}\frac{1}{(z+n)^2}$, where ζ is the Riemann zeta function. It follows that $\Psi'(s)$

> 0, hence $\Psi(s)$ is strictly increasing. Owing to the fact that $\Psi\left(\frac{2-\lambda}{pq}\right) = 0$, $\Psi(s) > 0$, when $S > \frac{2-\lambda}{pq}$. This shows that $\Phi'(s) > 0$. Similarly, we have $\Phi'(s) < 0$ when $s < \frac{2-\lambda}{pq}$. Consequently, the minimum of $\Phi(s)$ is

$$\Phi\left(\frac{2-\lambda}{pq}\right) = \left(B\left(\lambda - \left(1 - \frac{2-\lambda}{q}\right), 1 - \frac{2-\lambda}{q}\right)\right)^{1/p} \left(B\left(\lambda - \left(1 - \frac{2-\lambda}{p}\right), 1 - \frac{2-\lambda}{p}\right)\right)^{1/q}.$$

Since $1 - \frac{2-\lambda}{q} = \lambda - \left(1 - \frac{2-\lambda}{p}\right)$, $1 - \frac{2-\lambda}{p} = \lambda - \left(1 - \frac{2-\lambda}{q}\right)$

and $B(m, n) = B(n, m)$, we have the relation :

$$B\left(\lambda - \left(1 - \frac{2-\lambda}{q}\right), 1 - \frac{2-\lambda}{q}\right) = B\left(\lambda - \left(1 - \frac{2-\lambda}{p}\right), 1 - \frac{2-\lambda}{p}\right)$$

We therefore obtain $\Phi\left(\frac{2-\lambda}{pq}\right) = B^*$. The Lemma is proved.

3. MAIN RESULTS

In this section we will mainly establish new inequalities related to Fibonacci sequence.

Theorem 3.1. *Let* $a_n \geq 0$, $b_n \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $p \geq q > 1$, $1 - \frac{q}{p} < \lambda \leq 2$, $u(x)$ *and* $v(x)$ *be functions defined respectively by (1.4) and (1.5). If* $\sum_{m=1}^{\infty}\left\{(u(m))^{1-\lambda}(v(m))^{1-p}\right\}a_m^p < +\infty$ *and* $\sum_{n=1}^{\infty}\left\{(v(n))^{1-\lambda}(u(n))^{l-q}\right\}b_n^q < +\infty$, *then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \leq \frac{B^*}{\ln\frac{\sqrt{5}+1}{2}}\left\{\sum_{m=1}^{\infty}\{(u(m))^{l-\lambda}(v(m))^{l-p}\}a_m^p\right\}^{1/p}$$
$$\times\left\{\sum_{n=1}^{\infty}\{(v(n))^{l-\lambda}(u(n))^{l-q}\}b_n^q\right\}^{1/q} \tag{3.1}$$

where the constant factor $B^{*}/\ln\frac{\sqrt{5}+1}{2}$ is best possible. And the equality in (3.1) holds if and only if $\{a_n\}$, or $\{b_n\}$, is a zero-sequence.

Proof. Let's introduce into a parameter s such that $0 \le ps < 1$, By Lemma 2.1, $u(x)$ and $v(x)$ are derivable at point $x = n(n \in N)$. For convenience, we denote that $a_m = A_m(u'(m))^{1/q}$ and $b_n = B_n(v'(n))^{1/p}$, and then define two functions :

$$\alpha = \frac{A_m\{v'(n)\}^{1/p}}{(u(m)+v(n))^{\lambda/p}}\left(\frac{u(m)}{v(n)}\right)^{s} \text{ and } \beta = \frac{B_n\{u'(m)\}^{1/q}}{(u(m)+v(n))^{\lambda/q}}\left(\frac{v(n)}{u(m)}\right)^{1/s}. \tag{3.2}$$

Apply Hölder's inequality to estimate the right-hand side of (3.1) as follows :

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_m(v'(n))^{1/p}}{(u(m)+v(n))^{\lambda/p}}\left(\frac{u(m)}{v(n)}\right)^{s}\frac{B_n(u'(m))^{1/q}}{(u(m)+v(n)^{\lambda/q}}\left(\frac{v(n)}{u(m)}\right)^{s}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha\beta \le \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\alpha^{p}\right\}^{1/p}\left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\beta^{q}\right\}^{1/q}$$

$$= \left(\sum_{m=1}^{\infty}\omega_p(\lambda,m)A_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}\omega_q(\lambda,n)B_n^q\right)^{1/q} \tag{3.3}$$

Where $\omega_p(\lambda, m) = \sum_{n=1}^{\infty}\frac{v'(n)}{(u(m)+v(n))^{\lambda}}\left(\frac{u(m)}{v(n)}\right)^{ps}$

and $\omega_q(\lambda, n) = \sum_{m=1}^{\infty}\frac{u'(m)}{(u(m)+v(n))^{\lambda}}\left(\frac{v(n)}{u(m)}\right)^{qs}$

Notice that $v(0) = \frac{1}{\sqrt{5}}$, by Lemma 2.1, we have

$$\omega_p(\lambda, m) \le \int_0^{\infty}\frac{v'(x)}{(u(m)+v(x))^{\lambda}}\left(\frac{u(m)}{v(x)}\right)^{ps}dx = \int_0^{\infty}\frac{(u(m))^{-\lambda}(v'(x))}{(1+v(x)/u(m))^{\lambda}}\left(\frac{u(m)}{v(x)}\right)^{ps}dx$$

$$= \int_{v(0)/u(m)}^{\infty}\frac{(u(m))^{1-\lambda}}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{ps}dt = \int_{1/(\sqrt{5}u(m))}^{\infty}\frac{(u(m))^{1-\lambda}}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{ps}dt$$

$$= (u(m))^{1-\lambda}\left\{\int_0^{\infty}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{ps}dt - \int_0^{1/(\sqrt{5}u(m))}\frac{1}{(1+t)^{\lambda}}\left(\frac{1}{t}\right)^{ps}\right\}$$

$$\leq (u(m))^{1-\lambda} B(\lambda - (1-ps),\ 1-ps). \tag{3.4}$$

Simimarly, we have $\omega_q(\lambda, n) \leq (v(n))^{1-\lambda} B(\lambda - (1-qs),\ 1-qs)$. (3.5)

It follows from (3.3), (3.4) and (3.5) that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \leq \Phi(s)\left(\sum_{m=1}^{\infty}(u(m))^{1-\lambda}A_m^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(v(n))^{1-\lambda}B_m^q\right)^{1/q} \tag{3.6}$$

where $\Phi(s)$ is defined by (2.2).

It follows from Lemma 2.3 that the minimum of $\Phi(s)$ is B^* when $S = \dfrac{2-\lambda}{pq}$, where λ satisfies the constraint $1 - \dfrac{q}{p} < \lambda \leq 2$. Notice that $A_m^p = (u'(m))^{1-p} a_m^p$ and $B_n^q = (v'(n))^{1-q} b_n^q$. Therefore we obtain from (3.6) that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \leq B' \left\{\sum_{m=1}^{\infty}\{(u(m))^{1-\lambda}(u'(m))^{1-p}\}a_m^p\right\}^{1/p}$$

$$\times\left\{\sum_{n=1}^{\infty}\{(v(n))^{1-\lambda}(v'(n))^{1-q}\}b_n^q\right\}^{1/q} \tag{3.7}$$

And it is obvious that the equality in (3.7) if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence. It remains to need to show that the constant factor B^* in (3.7) is best possible.

Let $\tilde{a}_m = (u(m))^{-(2-\lambda+\varepsilon)/p}(u'(m))$ and $\tilde{b}_n = (v(n))^{-(2-\lambda+\varepsilon)/q}(v'(n))$. Assume that $0 < \varepsilon < (\lambda - 1) + \dfrac{q}{2p}$, then

$$\frac{1}{\varepsilon} = \int_1^{\infty} u^{-1-\varepsilon}du < \sum_{m=1}^{\infty}(u(m))^{-1-\varepsilon}(u'(m)) = \sum_{m=1}^{\infty}(u(m))^{1-\lambda}(u'(m))^{1-p}\tilde{a}_m^p$$

$$= (u(1))^{-1-\varepsilon}\ u'(1) + \sum_{m=2}^{\infty}(u(m))^{-1-\varepsilon}(u'(m))$$

$$< (u(1))^{-1-\varepsilon}\ u'(1) + \int_1^{+\infty} u^{-1-\varepsilon}du = \frac{3}{\sqrt{5}}\ln\frac{\sqrt{5}+1}{2} + \frac{1}{\varepsilon}$$

where $u(1) = 1$, $u'(1) = \frac{3}{\sqrt{5}}\ln\frac{\sqrt{5}+1}{2}$, (cf. (1.4), (1.5) and (2.1)).

Similarly, according to (1.4), (1.5) and (2.1), we have

$$\frac{1}{\varepsilon} < \sum_{n=1}^{\infty}(v(n))^{1-\lambda}(v'(n))^{1-q}\tilde{b}_n^q < (v(1))^{-1-\varepsilon}\ v'(1) + \frac{1}{\varepsilon} = \left(\frac{3}{\sqrt{5}}\right)^{-1-\varepsilon}\left(\ln\frac{\sqrt{5}+1}{2}\right) + \frac{1}{\varepsilon}$$

Hence $\sum_{m=1}^{\infty}(u(m))^{1-\lambda}\ (u'(m))^{1-p}\ \tilde{a}_m^p = \frac{1}{\varepsilon} + O(1)\ (\varepsilon \to 0)$

Similarly, we have $\sum_{n=1}^{\infty}(v(n))^{1-\lambda}\ (v'(n))^{1-q}\ \tilde{b}_n^q = \frac{1}{\varepsilon} + O(1).\ (\varepsilon \to 0)$

If B^* is not best possible, then there exists $k > 0$ and k less than B^* such that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(u(m)+v(n))^{\lambda}} < k\left(\sum_{n=1}^{\infty}(u(n))^{1-\lambda}(u'(n))^{1-p}\tilde{a}_n^p\right)^{1/p}\left(\sum_{n=1}^{\infty}(v(n))^{1-\lambda}(v'(n))^{1-q}\tilde{b}_n^q\right)^{1/q}$$

$$= \frac{1}{\varepsilon}(k + \circ(1))\ \ (\varepsilon \to 0) \tag{3.8}$$

On the other hand, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\tilde{a}_m\tilde{b}_n}{(u(m)+v(n))^{\lambda}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(u(m))^{-(2-\lambda+\varepsilon)/p}(u'(m))\,(v(n))^{-(2-\lambda+\varepsilon)/q}(v'(n))}{(u(m)+v(n))^{\lambda}}$$

$$> \int_1^{\infty}\int_1^{\infty}\frac{(u(x))^{-(2-\lambda+\varepsilon)/p}(v(y))^{-(2-\lambda+\varepsilon)/q}}{(u(x)+v(y))^{\lambda}}\ dudv$$

$$= \int_1^\infty \left\{ \int_1^\infty \frac{v^{-(2-\lambda+\varepsilon)/q}}{(u+v)^\lambda} dv \right\} u^{-(2-\lambda+\varepsilon)/p} du$$

$$= \int_1^\infty \left\{ \int_{v(1)/u(x)}^\infty \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt \right\} u^{-1-\varepsilon} du$$

$$= \frac{1}{\varepsilon} \int_{3/(\sqrt{5}u(x))}^\infty \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt.$$

where $v(1) = \dfrac{3}{\sqrt{5}}$. If the lower limit $\dfrac{3}{\sqrt{5}u(x)}$ of the integral is replaced by zero, then the resulting error is smaller than $(u(x))^{-\alpha}/\alpha$, where α is positive and independent of ε. In fact, we have

$$\int_0^{3/(\sqrt{5}u(x))} \frac{1}{(1+t)^\lambda} \left(\frac{1}{t}\right)^{(2-\lambda+\varepsilon)/q} dt < \int_0^{3/(\sqrt{5}u(x))} t^{(2-\lambda+\varepsilon)/q} dt = \frac{1}{\beta} \left(\frac{\sqrt{5}u(x)}{3}\right)^{-\beta}$$

where $\beta = 1 - (2 - \lambda + \varepsilon)/q$. If $0 < \varepsilon < (\lambda - 1) + \dfrac{q}{2p}$, then we may take α such that

$$\alpha = 1 - \frac{(2-\lambda) + (\lambda - 1) + q/2p)}{q} = \frac{1}{2p}$$

Consequently, we get

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\tilde{a}_m \tilde{b}_n}{(u(m) + v(n))^\lambda} > \frac{1}{\varepsilon}\{B^* + o(1)\} \quad (\varepsilon \to 0) \tag{3.9}$$

Clearly, when ε is small enough, the inequality (3.8) is in contradiction with (3.9). Therefore, B^* is the best possible value of which the inequality (3.7) keeps valid.

At last, by substituting the relation (2.1) into (3.7), the inequality (3.1) is yielded after simplifications. Clearly, the constant factor $B^*/\ln\dfrac{\sqrt{5}+1}{2}$ is best possible.

Thus the proof of Theorem is completed.

G-146087

If the denominator $(u(m)+v(n))^{\lambda}$ of the function of the left-hand side of (3.1) is replaced by $(u(m)+u(n))^{\lambda}$, then the following result is attained.

Theorem 3.2. With the assumptions as Theorem 3.1, if $\sum_{m=1}^{\infty}\left\{(u(m))^{1-\lambda}(v(m)^{1-p}\right\}a_m^p<+\infty$ and $\sum_{n=1}^{\infty}\left\{(u(n))^{1-\lambda}(v(n)^{1-q}\right\}b_n^p<+\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+u(n))^{\lambda}} \le \frac{B^{*}}{\ln\frac{\sqrt{5}+1}{2}}\left\{\sum_{m=1}^{\infty}\{(u(m))^{1-\lambda}(v(m))^{1-p}\}a_m^p\right\}^{1/p}$$

$$\times\left\{\sum_{n=1}^{\infty}\{(u(n))^{1-\lambda}(v(n))^{1-q}\}b_n^q\right\}^{1/q} \tag{3.10}$$

where the constant factor $B^{*}/\ln\frac{\sqrt{5}+1}{2}$ is best possible. And the equality in (3.10) holds if and only if $\{a_n\}$, or $\{b_n\}$ is a zero-sequence.

When $\lambda=1$, an extension on (1.3) is obtained basing on (3.1).

Corollary 3.3. *Let* $a_n\ge 0$, $b_n\ge 0$, $\frac{1}{p}+\frac{1}{q}=1$, $p\ge q>1$, $1-\frac{q}{p}<\lambda\le 2$, $u(x)$ and $v(x)$ be functions defined respectively by (1.4) and (1.5).

If $\sum_{m=1}^{\infty}(v(m))^{1-p}a_m^p<+\infty$ and $\sum_{n=1}^{\infty}(u(n))^{1-q}b_n^q<+\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \le \frac{\pi\csc\pi/p}{\ln\frac{\sqrt{5}+1}{2}}\left\{\sum_{m=1}^{\infty}(v(m))^{1-p}a_m^p\right\}^{1/p}\left\{\sum_{n=1}^{\infty}(u(n))^{1-q}b_n^q\right\}^{1/q} \tag{3.11}$$

where the constant factor $\dfrac{\pi\csc\pi/p}{\ln\frac{\sqrt{5}+1}{2}}$ is best possible.

In particular, for case $p=2$, an extension on Hilbert's inequality is gotten.

Corollary 3.4. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers* $0 < \lambda \le 2$, $u(x)$ *and* $v(x)$ *be functions defined respectively by (1.4) and (1.5).*

If $\sum\limits_{m=1}^{\infty} \left\{(v(m))^{1-\lambda}(v(m))^{-1}\right\} a_m^2 < +\infty$ and $\sum\limits_{n=1}^{\infty}\left\{(v(n))^{1-\lambda}(v(n))^{-1}\right\} b_n^2 < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \le \frac{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{\ln\frac{\sqrt{5}+1}{2}}\left\{\sum_{m=1}^{\infty}\left\{(u(m))^{1-\lambda}(v(m))^{-1}\right\} a_m^2\right\}^{1/2}$$

$$\times\left\{\sum_{n=1}^{\infty}\left\{(v(n))^{1-\lambda}(u(n))^{-1}\right\} b_n^2\right\}^{1/2} \tag{3.12}$$

where the constant factor $\dfrac{B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{\ln\frac{\sqrt{5}+1}{2}}$ is best possible, here $B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ is beta function.

When $\lambda = 1$ and $p = 2$, another extension on Hilbert's inequality is attained.

Corollary 3.5. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be sequences of real numbers.* $u(x)$ *and* $v(x)$ *be functions defined respectively by (1.4) and (1.5).*

If $\sum\limits_{m=1}^{\infty}(v(m))^{-1}a_m^2 < +\infty$ and $\sum\limits_{n=1}^{\infty}(u(n))^{-1}b_n^2 < +\infty$, then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(u(m)+v(n))^{\lambda}} \le \frac{\pi}{\ln\frac{\sqrt{5}+1}{2}}\left\{\sum_{m=1}^{\infty}(v(m)^{-1}a_m^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}(u(n))^{-1}b_n^2\right\}^{1/2} \tag{3.13}$$

where the constant factor $\dfrac{\pi}{\ln\frac{\sqrt{5}+1}{2}}$ is best possible,

REFERENCES

1. G. H. Hardy, J. E. Littlewood, G. Polya. Inequalities, Cambridge Unive. Press, Cambridge, 1952.
2. D. S. Mitrinovic, J. E. Pecaric, A. M. Fink, Inequalities Involving Functions and Their Integral and Derivatives, Kluwer Academic, Boston, 1991.
3. Yang Bicheng and Lokenath Debnath, On the Extended Hardy-Hilbert's Inequality, J. Math. Anal. Appl. Vol. 272, 1(2002), 187-199.
4. Yang Bicheng, On an Extension of Hardy-Hilbert's Inequality, Chinese Ann. Math. (Ser. A), Vol. 23, 2(2002), 247-254.
5. Yang Bicheng, A Note on Hilbert's integral Inequality, Chinese Quarterly Journal of Mathematics, Vol. 13, 4(1998), 83-86.
6. Yang Bicheng, On Hilbert's Integral Inequality, J. Math. Anal. Appl. Vol. 220, 2(1998), 778-785.
7. Kuang Jichang, On New Extensions of Hilbert's Integral Inequality, J. Math. Anal. Appl., Vol. 235, 2(1999), 608-614.
8. Yang Bicheng, On a General Hardy-Hilbert's Inequality With a Best Value, Chinese Ann. Math. (Ser. A), Vol. 21, 4(2000), 401-408.
9. Wang Zhuqi and Guo Dongren, An Introduction to Special Functions, Science Press, Beijing, 1979.

Department of Mathematics and Computer Science,
Normal College, Jishou University,
Jishou Human 416000,
People's Republic of China.)
E-mail : mingzhegao@163.com

Department of Mathematics,
University of Texas-Pan American,
Edinburg, TX 78539, USA
E-mail : debnath1@panam.edu

Jour. Pure Math., Vol. 21, 2004, pp. 25-30

AN EXTENSION OF THE FUGLEDE-PUTNAM THEOREM TO *P*-HYPONORMAL OPERATORS[†‡]

SALAH MECHERI[1]

ABSTRACT : The familiar Fuglede-Putnam Theorem is as follows (see [3], [7] and [8]): If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$. In this paper, the hypothesis on A and B can be relaxed by using a Hilbert-Schmidt operator X : Let A be p-hyponormal and B^* be invertible p-hyponormal such that $AX = XB$ for a Hilbert Schmidt operators X. Then $A^*X = XB^*$. As consequence of this result, we obtain that the range of the generalized derivation induced by this class of operators is orthogonal to its kernel.

Key words : p-hyponormal operator, Hilbert Shmidt class, orthogonality. Fuglede-Putnam theorem.

1. INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded operators on H. For any operator A in $B(H)$ set, as usual, $|A| = (A^*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions : A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-hyponormal $(0 < p \leq 1)$ if $(|A|^{2p} - |A^*|^{2p}) \geq 0$, and normaloid if $\|A\| = r(A)$ (the spectral radius of A). Let (N), (HN), (PHN), and (NL) denote the classes constituting of normal, hyponormal, p-hyponormal, and normaloid operators. These classes are related by proper inclusion :

$$(N) \subset (HN) \subset (PHN) \subset (NL).$$

Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators and also p-hyponormal operator have been recently studied by many authors (see [1], [6], [16], and [15]). The familiar Fvglede-Putnam theorem is as follows (see [3], [7] and [8]) :

[†]2000 Mathematics Subject Classification : Primary 47B47, 47A30, 47B20; Secondary 47B10.

[‡]This work is supported, by the research renter project No. Math/2005/04.

[1]King Saud University, College of Science, Department of Mathematics, P.O. Box 2455, Riyadh 11451, Saudi Arabia, e-mail : mecherisalah@hotmail.com

Theorem 1.1 *If A and B are normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

S. K. Berberian [2] relaxes the hypothesis on A and B in Theorem 1.1 as the cost of requiring X to be Hilbert-Schmidt class. H. K. Cha [4] showed that the hyponormality in the result of Berberian [2] can be replaced by the quasihypo-normality of A and B^* under some additional conditions. In this paper we will show that the quasihyponormality can be replaced by the p-hyponormality of A and B^*. Let $T \in B(H)$ be compact, and let $s_1(T) \geq s_2(T) \geq \dots \geq 0$ denote the singular values of T, i.e., the eigenvalues of $|T| = (T^*T)^{\frac{1}{2}}$ arranged in their decreasing order. The operator T is said to belong to the Schatten p-class C_p if $\| T \|_p = \left[\sum_{i=1}^{\infty} s_j(T)^p\right]^{\frac{1}{p}} = \left[tr|T|^p\right]^{\frac{1}{p}} < \infty$, $1 \leq p < \infty$, where tr denotes the trace functional. Hence $C_1(H)$ is the trace class, $C_2(H)$ is the Hilbert-Schmidt class, and C_∞ is the class of compact operators with $\| T \|_\infty = s_1(T) = \sup_{\|f\|=1} \| Tf \|$ denoting the usual operator norm, For the general theory of the Schatten p-classes the reader is referred to [12], [13]. Let $\delta_{A,B}$ be the generalized derivation defined on $B(H)$ by $\delta_{A,B}(X) = AX - XB$. It is clear that $\delta_{A,B}(C_p) \subseteq C_p$. However it can also happen that $\delta_{A,B}(X) \in C_p$ for some $X \in B(H) \setminus C_p$, hence $ran(\delta_{A,B}|C_p) \subseteq ran\,\delta_{A,B} \cap C_p$ and then we also have $\overline{ran\left(\delta_{A,B}|C_p\right)}^{C_p} \subseteq \overline{ran\,\delta_{A,B} \cap C_p}^{C_p}$, where $\overline{(.)}^{C_p}$ denotes the closure of the C_p norm. A. Turnsek [14] asked the following question : When the reverse inclusion is possible? In this note we consider the question when

$$\overline{ran(\delta_{A,B}|_{C_2})}^{C_2} = \overline{ran\delta_{A,B} \cap C_2}^{C_2} \tag{1.1}$$

Or equivalently, if $\delta_{A,B}(X) \in C_2$ then $\delta_{A,B}(X) = Lim_n\, \delta_{A,B}(X_n)$, and $X_n \in C_2$. We prove that this holds in the case when A is p-hyponormal operator and B^* is invertible p-hyponormal operator.

2. MAIN RESULTS

Lemma 2.1 *Let A and B be operators in $B(H)$. If A and B^* are p-hyponormal operators, then the operator $\tau : C_2H \to C_2(H)$ defined by $\tau X = AXB$ is p-hyponormal.*

Proof. It is known [2] that $\tau^*X = A^* XB^*$. Note that by the uniqueness of the square root of a positive operators we have

$$(\tau^*\tau)^{\frac{1}{2}}X = |\tau|X = |A|X|B^*|,\ (\tau\tau^*)^{\frac{1}{2}}X = |\tau^*|X = |A^*|X|B|.$$

Thus

$$|\tau|^{2p}X - |\tau^*|^{2p} X = (|\tau|^{2p} - |\tau^*|^{2p})X = |A|^{2p}X|B^*|^{2p} - |A^*|^{2p}X|B|^{2p} =$$
$$(|A|^{2p} - |A^*|^{2p})X|B^*|^{2p} + |A^*|^{2p}X(|B^*|^{2p} - |B|^{2p}).$$

Since A and B^* are p-hyponormal operators, we have

$$(|\tau| - |\tau^*|) \geq 0.$$

The following theorem is well known. □

Theorem 2.1 *Let A be p-hyponormal operator. if* $\lambda \in \sigma_p(A) \setminus \{0\}$, *then* $\lambda \in \sigma_p(A^*))$ *for all* $X \in H$ *and for all* $\lambda \in \mathbb{C}$.

Proof. It is known that the class of p-hyponormal operators is included in the class of normaloid operators. It is also known that if A is normaloid, then the nonzero eigenvalues of A are normal eigenvalues (i.e., if $\lambda \in \sigma_p(A) \setminus \{0\}$, then $\lambda \in \sigma_p(A^*)$). Which complete the proof. □

Now we are ready to extend Putnam-Fuglede theorem to p- hyponormal operators.

Theorem 2.2 *Let A be p-hyponormal operator and* B^* *be an invertible p-hyponormal operator such that* $AX = XB$ *for* $X \in C_2(H)$. *Then* $A^* X = XB^*$.

Proof. Let $\mathcal{K}$ be defined on $C_2(H)$ by $\mathcal{K}Y = AYB^{-1}$ for all $Y \in C_2(H)$. Since B^* is p-hyponormal, $(B^*)^{-1}$ is also p-hyponormal (see [9]). Then it follows from Lemma 2.1 that $\mathcal{K}$ is invertible p-hyponormal, furthermore, $\mathcal{K}X = AXB^{-1} = X$ and so, X is an eigenvector of $\mathcal{K}$. Now by applying Theorem 2.1 we get $\mathcal{K}^* X = A^* X(B^{-1})^* = X$, that is, $A^* X = XB^*$ and the proof is achieved. □

As a consequence of the above theorem, we obtain

Corollary 2.1 [2] *Assume that A,* B^* *and X are operators in an Hilbert space H such that A,* B^* *are hyponormal operators and* $AX = X B$. *Assume also that X is an operator of Hilbert-Schmidt class. Then* $A^*X = XB^*$ *under either of the following hypothesis*

1. *A and* B^* *are hyponormal;*
2. *B is invertible and* $\|A\|.\|B^{-1}\| \leq 1$.

Proof. 1. Is a simple consequence of the above theorem.

2 The result of Y. Tong [17] guarantees that the above condition implies that for all $T \in ker(\delta_{A,B} \mid \mathcal{K}(H))$, $\overline{R(T)}$ reduces A, $\ker(T)^{\perp}$ reduces B, and $A\big|_{\overline{R(T)}}$ and $B\big|_{\ker(T)^{\perp}}$ are unitary operators. Take $\mathcal{H}_1 = H = \overline{ranS} \oplus \overline{ranS}^{\perp}$, $H_2 = H = \ker S \oplus \ker S^{\perp}$. According to the decomposition of $\mathcal{K}$ and for $A_1 : \mathcal{H}_1 \to H_1$, $A_2 : H_2 \to H_2$, $S : H_2 \to H_1$, we can write

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \; B^* = \begin{pmatrix} B_1^* & 0 \\ 0 & B_2^* \end{pmatrix}, \; S = \begin{pmatrix} S_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

From $AS = SB$ it follows that $A_1S = SB_1$ and since A_1, B_1 are unitary operators we obtain $A_1^*S = SB_1^*$ and the result holds by the above theorem. The above inequality holds in particular if $A = B$ is isometric, in other words $\|Ax\| = \|x\|$ for all $x \in H$.

Theorem 2.3 [11] *Let A, B be operators in B(H) and* $S \in C_2$. *Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2 \tag{2.1}$$

and

$$\|\delta^*_{A,B}(X) + S\|_2^2 = \|\delta^*_{A,B}(X)\|_2^2 + \|S\|_2^2 \tag{2.2}$$

if and only if $\delta_{A,B}(S) = 0 = \delta_{A^*,B^*}(S)$, *for all* $X \in C_2(H)$.

Corollary 2.2 *Let A, B be operators in B(H) and* $S \in C2$. *Then*

$$\|\delta_{A,B}(X) + S\|_2^2 = \|\delta_{A,B}(X)\|_2^2 + \|S\|_2^2$$

and

$$\|\delta^*_{A,B}(X) + S\|_2^2 = \|\delta^*_{A,B}(X)\|_2^2 + \|S\|_2^2$$

if and only if either of the following hypothesis hold :

(1) *A and* B^* *hyponormal operators*

(2) *B is invertible and* $\|A\|.\|B^{-1}\| \leq 1$.

(3) $A, B \in B(H)$ *such that* $\|Ax\| \geq \|x\| \geq \|Bx\|$ *for all* $x \in \mathcal{H}$

(4) *A is p-hyponormal and* B^* *is invertible p-hyponormal.*

(5) *A is k-quasihyponormal and* B^* *is invertible k-quasihyponormal.*

Proof. Concerning (1) and (2) it suffices to apply corollary 2.1. For (3) In this case it suffices to take $A_1 = \|B\|^{-1} A$ and $B_1 = \|B\|^{-1} B$, then $\|A_1x\| \geq \|x\| \geq \|B_1x\|$ and the result holds by (2) for all $x \in H$. The case (4) follows from Theorem 2.2 and the case (5) follows by ([11], Theorem 2.2)

Now we will answer the question when

$$\overline{ran(\delta_{A,B} \setminus_{C_2})}^{C_2} = \overline{ran\delta_{A,B \cap C_2}}^{C_2}.$$

Theorem 2.4 [11] *Let A and B be operators in B(H) such that* $Ker\delta_{A,B} \subseteq Ker\delta_{A^*,B^*}$. *Then*

$$\overline{ran(\delta_{A,B} \backslash_{C_2}(H))}^{C_2(H)} = \overline{ran\delta_{A,B} \cap_{C_2(H)}}^{C_2(H)}.$$

Corollary 2.3 *Let A and B be operators in B(H). Then*

$$\overline{ran(\delta_{A,B} \backslash_{C_2(H)})}^{C_2(H)} = \overline{ran\delta_{A,B} \cap_{C_2(H)}}^{C_2(H)}$$

under either of the hypothesis (1) *to* (5) *in the above corollary.*

REFERENCES

1. A. Aluthge, On p-hyponormal operators for $0 < p < 1$, J. Integral equation and operator theory, 13 (1990), 307-315.

2. S. K. Berberian, Extension of a theorem of Fuglede and Putnam, Proc. Amer. Math. Soc., 71 (1978), 113-114.

3. J. B. Conway, Subnormal operators, Research notes in Math., Pitnam Advanced Pub. Program, 51 (1981).

4. H. K. Cha, An extension of Fuglede-Putnam theorem to quasihyponormal operators using a Hilbert-Schmidt operator, Youngnam Math. J., 1(1994), 73- 76.

5. S. L. Campell and B.C.Gupta, On k-quasihyponormal operators, Math. Japonica., 232(1978), 185-189.

6. M. Cho and T. Huruya, p-hyponormal operators, $0 < p < \frac{1}{2}$, Comment.Math., 33(1993), 23-29.

7. T. Furuta, Invitation to linear operators, Taylor Francis Inc., (2001).

8. P. R. Halmos, A Hilbert space problem book, spring-Verlag, New York, (1974).

9. M. Y. Lee and S. H. Lee, An extension of the Fuglede-Putnam theorem to p-quasihyponormal operator, Bull. Korean Math. Soc., 35(1998), 319-324.

10. M. Y. Lee, An extension of the Fuglede-Putnam theorem to (p, k)-quasihyponormal operator, Kyungpook Math. J., 44(2004), 593-596.

11. S. Mecheri, An extension of the Fuglede-Putnam theorem to k-quasihyponormal operator, submitted.

12. R. Schatten, Norm ideals of completely continuous operators, Springer-Verlag, Berlin, 1960.

13. B. Simon, Trace ideals and their applications, Cambridge. Univ. Press, Cambridge, UK, 1979.

14. A. Turnsek, On the range of elementary operators, Pu Publ. Math. Debrecen, 63(2003), n3, 293-304.

15. H Tadashi, A note on p-hyponormal operators., Proc. Amer. Math. Soc, 125(1997), 221-230.

16. A. Uchiyama, Berger-Shaws theorem for p-hyponormal operators., Inequalities in operator theory and related topics (Japanese) (Kyoto, 1997). S ū rikaisekikenky ū sho K ō ky ū roku No. 1027, (1998), 110-121.

17. Y. Tong, Kernels of generalized derivations, Acta. Sci. Math. 54 102(1990), 159-169.

Jour. Pure Math., Vol. 21, 2004, pp. 31-38

A GENERALIZATION OF FUGLEDE-PUTNAM THEOREM

SALAH MECHERI

ABSTRACT : The equation $AX = XB$ implies $A^*X = XB^*$ when A and B are normal is known as the familiar Fuglede-Putnam theorem. In this paper we show that if A is a dominant operator and B^* is a log-hyponormal operator or p-hyponormal operator such that $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$. Also, it is shown that if A is a (p, k)-quasihyponormal operator and B^* is a log-hyponormal operator or p-hyponormal operator such that $AX = XB$ for some $X \in B(H)$, then $A^*X = XB^*$.

1991 Mathematics Subject Classification : Primary 47B47, 47A30, 47B20; Secondary 47B10.

Key words and phrases : Fuglede-Putnam theorem, Dominant operator, log-hyponormal operator, hyponormal operator, (p, k)-quasihyponormal operator.

1. INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded operators on H. For any operator A in $B(H)$ set, as usual, $|A| = (A*A)^{\frac{1}{2}}$ and $[A^*, A] = A^*A - AA^* = |A|^2 - |A^*|^2$ (the self commutator of A), and consider the following standard definitions : A is normal if $A^*A = AA^*$, hyponormal if $A^*A - AA^* \geq 0$, p-hyponormal if $(|A|^{2p} - |A^*|^{2p}) \geq 0$, $(0 < p \leq 1)$. A is said to be log-hyponormal if A is invertible and satisfies the following inequality

$$log(A^* A) \geq log(AA^*).$$

It is known that invertible p-hyponormal operators are log-hyponormal operators but the converse is not true [18]. However it is very interesting that we may regards log-hyponormal operators as 0-hyponormal operators [18, 19]. The idea of log-hyponormal operator is due to Ando [2] and the first paper in which log-hyponormality appeared is [11]. See [1, 18, 19, 20] for properties of log-hyponormal operators.

A is said to be p-quasihyponormal if $A^*((A^*A)^p - (AA^*)^p)A \geq 0$ $(0 < p \leq 1)$, (p, k)-quasihyponormal if $A^{*k}((A^*A)^p - (AA^*)^p)A^k \geq 0$ $(0 < p \leq 1, k \in \mathbf{N})$, if $p = 1$, $k = 1$ and $p = k = 1$, then A is k-quasihyponormal, p-quasihyponormal and quasihyponormal respectively. A is normaloid if $||A|| = r(A)$ (the spectral radius of A). Let (N), (HN), $Q(p)$, $(Q(p, k))$ and

(NL) denote the classes constiting of normal, hyponormal, p-quasihyponormal, (p, k)-quasihyponormal, and normaloid operators. These classes are related by proper inclusion relations

$$(N) \subset (HN) \subset (Q(p)) \subset (Q(p, k)) \subset (NL).$$

An operator $A \in B(H)$ is called *dominant* by J. G. Stampfli and B. L. Wadhwa [16] if, for all complex λ, $range(A - \lambda) \subseteq range(A - \lambda)^*$, or equivalently, if there is a real number $M_\lambda \geq 1$ such that $\|(A - \lambda)^* f\| \leq M_\lambda \|(A - \lambda) f\|$, for all $f \in H$. If there exists a real number M such that $M_\lambda \leq M$ for all λ, the dominant operator A is said to be M-hyponormal. A 1-hyponormal is hyponormal. Let (D) and $(m - H)$ denote the classes of dominant and M-hyponormal operators. Then

$$(N) \subset (H) \subset (m - H) \subset (D).$$

The familiar Fuglede-Putnam theorem is as follows (see [4], [10] and [12]) :

Theorem 1.1. *If A and B normal operators and if X is an operator such that $AX = XB$, then $A^*X = XB^*$.*

Many mathematicians have extented this theorem for several classes of operators for example (see [16], [21]).

2. MAIN RESULTS

We will recall some known results which will be used in the sequel.

Definition 2.1. *Given $A, B \in B(H)$. We say that the pair (A, B) has $(FP)_{B(H)}$ the Fuglede-Putnam property if $AC = CB$ for some $C \in B(H)$, implies $A^*C = CB^*$.*

Theorem 2.1. [8] *Let $A \in B(H)$ and $B \in B(K)$. Then the following assertions are equivalent*

(i) The (A, B) has the property $(FP)_{B(H, K)}$.

(ii) If $AC = CB$ for some $C \in B(H, K)$, then $\overline{ran(C)}$ reduces A, $(KerC)^\perp$ reduces and $A|_{\overline{ran(C)}}$ and $B|_{\overline{(KerC)^\perp}}$ are normal operators.

Lemma 2.1. [16] *Let A be dominant and let M be invariant subspace of A for which $A|_M$ is normal. Then M reduces A.*

Lemma 2.2. [20] *If A is* log*-hyponormal and M an invariant subspace of A for which $A|_M$ is normal, then M reduces A.*

Lemma 2.3. [22] *Let A be dominant operator and M be an invariant subspace of M. Then $A|_M$ is a dominant operator.*

Theorem 2.2. *Let A be dominant in $B(H)$ and B^* be* log*-hyponormal or p-hyponormal in $B(K)$ such that $AC = CB$ for some $C \in B(H, K)$. Then $A^*C = CB^*$.*

Proof. The case in which A is dominant and B^* is log-hyponormal is proved in [21]. Here we will give a different proof for *log*-hyponormal and p-hyponormal. Since $\overline{ranC}$ is invariant under A and $(KerC)^{\perp}$ is invariant under B^*, the operators A, B and C can be written on the following decompositions of H and K

$$K = (KerC)^{\perp} \oplus (KerC),\ H = \overline{ranC} \oplus \overline{ranC}^{\perp}$$

as follows :

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},\ B = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix},\ C = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} :$$

$$(ker\ C)^{\perp} \oplus (kerC) \to \overline{ranC} \oplus \overline{ranC}^{\perp}.$$

Thus from $AC = CB$ we obtain,

$$A_1C_1 = C_1B_1. \tag{2.2}$$

Let $B_1 = U|B_1|$ be the polar decomposition of B_1. We usually define the Aluthge transform of B_{11} by $\widetilde{B}_1 = |B_1|^{\frac{1}{2}}U|B_1|^{\frac{1}{2}}$. $\widetilde{B}_1 = V|\widetilde{B}_1|$, and define the second Aluthge transform of B_1 by $\hat{B}_1 = |\widetilde{B}_1|^{\frac{1}{2}}V|\widetilde{B}_1|^{\frac{1}{2}}$. Since $V|\widetilde{B}_1| = |\widetilde{B}_1^*|V$, the equality (2.2) becomes

$$A_1C_1 = C_1|\widetilde{B}_1^*|C. \tag{2.3}$$

Multiply the two members of (2.3) by $|\widetilde{B}_1^*|^{\frac{1}{2}}$ in right we get

$$A_1(C_1|\widetilde{B}_1^*|^{\frac{1}{2}}) = (C_1|\widetilde{B}_1^*|^{\frac{1}{2}})|\widetilde{B}_1^*|^{\frac{1}{2}}V\,|\widetilde{B}_1^*|^{\frac{1}{2}}.$$

Since the second Aluthge transform $\hat{B}_1^* = |\widetilde{B}_1^*|^{\frac{1}{2}}V^*|\widetilde{B}_1^*|^{\frac{1}{2}}$ is hyponormal [21] and A_1 is dominant by Lemma 2.1, the pair $(A_{11}, \hat{B}_{11}^*)$ has the Fuglede-Putnam property by [8]. Therefore the restrictions

$$A_1|_{\overline{ran(C_1|\widetilde{B}_1^*|^{\frac{1}{2}})}} \quad \text{and} \quad \widetilde{B}_1^*|_{[Ker(C_1|\widetilde{B}_1^*|^{\frac{1}{2}})]^{\perp}}$$

are normal operators by Theorem 2.1. Since C_1 is a one to one mapping with dense range and $|\widetilde{B}_1^*|^{\frac{1}{2}}$ is a one to one mapping, it follows that

$$\overline{ran(C_{11}|\widetilde{B}_1^*|^{\frac{1}{2}}|)} = \overline{ran(C_1)} = \overline{ran(C)}$$

and $$ker(C_1|\widetilde{B}_1^*|^{\frac{1}{2}}) = kerC_1 = kerC.$$

Hence $\hat{B}_1^*$ is normal by [16]. Therefore B_1 is normal by [17]. Since A_1 is domonant by Lemma 2.3 and the restriction A_1 is normal, $ran(C)$ reduces A Lemma 2.1, similarly, since B^* is log-hyponormal and the restriction B_1 is normal, $[KerC_1]^{\perp}$ reduces B_1^* by Lemma 2.2. Since the pair (A_1, B_1) has the Fuglede-Putnam property, $A_1^*C_1 = C_1B_1^*$. Which implies that $A^*C = CB^*$.

Now if A is dominant and B^* is p-hyponormal. We consider two cases :

Case 1. $(\frac{1}{2} < p \leq 1)$. It is known that the Aluthge transform B^* is hyponormal. Here we don't need the second Aluthge transform of B_{11}^*, but only the Aluthge transform of B_1^* and the proof is the same as the above proof.

Now concerning the case $2(0 < p \leq \frac{1}{2})$, it suffices to take $p' = p + \frac{1}{2}$, where $p' \in (\frac{1}{2}, 1]$. It comes back that $\hat{B}_1^*$ is p'-hyponormal and the proof can be achieved by the same way as above □

Corollary 2.1. *A is normal if and only if A is dominant and A^* is* log-*hyponormal or p-hyponormal.*

Proof. It suffices to put $A = B = C$ in (2.4). □

Lemma 2.4. [?] *Let $A \in B(H)$ be (p, k)-quasihyponormal. If M is an invariant subspace for A, then the restriction of A to M is (p, k)-quasihyponormal.*

Theorem 2.3. *Let $A, B \in B(H)$ be such that A is (p, k)-quasihyponormal with $\sigma_p(A)\backslash\{0\} \neq \phi$ and B^** log-*hyponormal or p-hyponormal. If $AC = CB$ for some $C \in B(H)$, then $A^*C = CB^*$.*

Proof. Let $0 \neq \lambda \in \sigma_p(A)$. Let's consider the following decompositions of H.

$$H = [Ker(A - \lambda I)]^{\perp} \oplus Ker(A - \lambda I) = (Ker\ B^*)^{\perp} \oplus Ker\ B^*.$$

Since $Ker(A - \lambda I)$ reduces A, we can write A, B and C as follows

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & \lambda I \end{bmatrix},\ B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix},\ C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}.$$

$$AC = \begin{bmatrix} A_1C_1 & A_1C_2 \\ \lambda C_3 & \lambda C_1 \end{bmatrix},\ CB = \begin{bmatrix} C_1B_1 & 0 \\ C_3B_1 & 0 \end{bmatrix}.$$

Hence

$AC = CB$ implies $A_1C_1 = C_1B_1$ and $A_1C_2 = \lambda C_4$, $\lambda C_3 = C_3B_1$.

Or, equivalently $A_1C_1 = C_1B_1$, $C_2 = C_4 = 0$, and $\lambda C_3 = C_3B_1$

Let

$$A_1C_1 = C_1B_1, \tag{2.6}$$

where A_1 is (p, k)-quasihyponormal and B_1^* is log-hyponormal. Let's consider the following decompositions

$[Ker(A - \lambda I)]^{\perp} = \overline{ran(C_1)} \oplus [ran(C_1]^{\perp}$, $(Ker\ B^*)^{\perp} = \overline{Ker(C_1)} \oplus [KerC_1]^{\perp}$.

From (2.6), $\overline{ran(C_1)}$ is invariant for A_1 and $KerC_1$ is invariant for B_1. Accordisng to Lemma (2.4) A_1 is (p, k)-quasihyponormal. Since $\widetilde{B}_1^*$ is injective, $\hat{B}_1^* = |\widetilde{B}_1^*|^{\frac{1}{2}} U^* |\widetilde{B}_1'|^{\frac{1}{2}}$ is also injective. Now by the same way as in the proof of the above theorem, we obtain

$$A_1^*C_1 = C_1B_1^*.$$

Hence

$$A^*C = \begin{bmatrix} A_1^*C_1 & 0 \\ \overline{\lambda} C_3 & 0 \end{bmatrix},\ CB^* = \begin{bmatrix} C_1B_1^* & 0 \\ C_3B_1^* & 0 \end{bmatrix}$$

Since

$$\lambda C_3 = C_3B_1,$$

λI dominant and B_1^* log-hyponormal, by applying the above theorem we get

$$\overline{\lambda}\, C_3 = C_3B_1^*.$$

Therefore $A^*C = CB^*$.

Now if A is (p, k)-quasihyponormal with $\sigma_p(A) \setminus \{0\} \neq \phi$ and B^* p-hyponormal. We consider two cases :

Case 1. $(\frac{1}{2} < p \leq 1)$. It is known that the Aluthge transform B^* is hyponormal. Here we don't need the second Aluthge transform of B_{11}^*, but only the Aluthge transform of B_{11}^* and the proof is the same as the above proof.

Now concerning the case $2(0 < p \leq \frac{1}{2})$, it suffices to take $p' = p + \frac{1}{2}$, where $p' \in (\frac{1}{2}, 1]$. It comes back that $\widetilde{B}_{11}^*$ is p'-hyponormal and the proof can be achieved by the same way as above Which completes the proof. □

As a consequences of the above theorem we obtain,

Corollary 2.2. A is normal if and only if A is (p, k)-quasihyponormal with $\sigma_p(A) \setminus \{0\} \neq \phi$ and B^* log-hyponormal.

Corollary 2.3. Let $A, B \in B(H)$ be such that A invertible (p, k)-quasihyponormal operator, log-hyponormal or p-hyponormal and B^* log-hyponormal, p-hyponormal or (p, k)-quasihyponormal inversible. If $AC = CB$ for some $C \in B(H)$, then $A^*C = CB^*$.

Proof. It is known that an invertible (p, k)-quasihyponormal operator is invertible p-hyponormal ([15], Lemma 3) and an invertible p-hyponormal is log-hyponormal [18]. Hence the result holds by ([21], Theorem 8). □

Corollary 2.4. Let $A, B \in B(H)$ be such that A is k-quasihyponormal with $\sigma_p(A) \setminus \{0\} \neq \phi$ and B^* log-hyponormal. If $AC = CB$ for some $C \in B(H)$, then $A^*C = CB^*$.

Corollary 2.5. Let $A, B \in B(H)$ be such that A is p-quasihyponormal with $\sigma_p(A) \setminus \{0\} \neq \phi$ and B^* log-hyponormal. If $AC = CB$ for some $C \in B(H)$, then $A^*C = CB^*$.

Acknowledgment. This research was supported by the K. S. U research center project no. Math/2005/04.

REFERENCES

1. A Aluthge and D. Wang, An operator inequality which implies paranormality, Math. Inequalities and Applications, 2(1999), 113-119.
2. T. Ando, Operators with a norm condition, Acta. Sci. Math(Szeged) 33(1972), 169-178.
3. S. K. Berberian, Extension of a theorem of Fuglede and Putnam, Proc. Amer. Math. Soc., 71(1978), 113-114.
4. J. B. Conway, Subnormal operators, Research notes in Math., Pitnam Advanced Pub. Program, 51(1981).
5. H. K. Cha, An extension of Fuglede-Putnam theorem to quasihyponormal operators using a Hilbert-Schmidt operator, Youngnam Math. J., 1(1994), 73-76.
6. S. L. Campell and B. C. Gupta, On k-quasihyponormal operators, Math. Japonica., 232(1978), 185-189.
7. M. Cho and T. Huruya, p-hyponormal operators, $0 < p < \frac{1}{2}$, Comment. Math., 33(1993), 23-29.
8. B. P. Duggal, On intertwining operators, Mh. Math., 106(1988), 139-148.
9. B. P. Duggal, On dominant operators, Arch. Math., 46(1986), 353-359.
10. T. Furuta, Invitation to linear operators, Taylor Francis Inc., (2001).
11. M. Fujii, C. Himeji and A. Matsumoto, Theorems of Ando and Saito for p-hyponormal operators, Math. Japonica 39(1994), 595-598.
12. P. R. Halmos, A Hilbert space problem book, spring-Verlag, New York, (1974).
13. E. Heinz, Beitrage zur Storungstheorie der Specktralzerlegung, Math. Ann., 123 (1951), 415-438.
14. K. Lowner, Uber monotone Matrixfunktionen., Math. Z, 38(1934), 1777-216.
15. M. Y. Lee, An extension of the Fuglede-Putnam theorem to (p, k)-quasihyponormal operator, Kyungpook Math. J., 44(2004), 593-596.
16. J. G. Stampfli and B. L. Wadhwa An asymmetric Putnam-Fuglede theorem for dominant operators, Indiana Univ. Math. J. 25(1976)., 359-365.
17. H. Tadashi, A note on p-hyponormal operators., Proc. Amer. Math. Soc, 125(1997), 221-230.
18. K. Tanahashi, On log-hyponormal operators, Integral equations Operator Theory., 34(1999), 364-372.

19. K. Tanahashi, Putnam's inequality for log-hyponormal operators, Integral equations Operator Theory, to appear.

20. A. Uchiyama and K. Tanahashi, Fuglede-Putnam's theorem for p-hyponormal or log-hyponormal operators, Glaskow Math. J., 44(2002), 397-410.

21. K Tanahashi, A. Uchiyama and Muneo Cho, Isolated points of spectrum of (p, k)-quasihyponormal operators, Linear Algebra Appl., 382(2004), 221-229.

22. I. H. Jeon, K. Tanahashi and A. Uchiyama, On Quasisimilarity for log-hyponormal operators, Glaskow Math. J., 46(2004), 169-176.

23. T. Yoshino, Remark on the Generalized Putnam-Fuglede theorem, Proc. Amer. Math. Soc., 95(1985), 571-572.

Department of Mathematics,
King Saud University, College of Science
P.O. Box 2455, Riyadh 11451, Saudi Arabia.
E-mail : mecherisalah@hotmail.com

Jour. Pure Math., Vol. 21, 2004, pp. 39-48

COMPOSITION OF ENTIRE AND MEROMORPHIC FUNCTIONS AND THEIR GROWTH PROPERTIES

SANJIB KUMAR DATTA

ABSTRACT : In the paper we study some growth properties of composite entire and meromorphic functions improving some earlier results. An example is given to show that a condition in a theorem of the paper is essential.

AMS subject classification (2000) : 30D35, 30D30.

Keywords and phrases : Entire function, Meromorphic function, Growth, Order, Hyper Order, Type.

1. INTRODUCTION AND DEFINITIONS

For any two transcendental entire functions f and g, Clunie [2] proved that $\lim\limits_{r\to\infty} \frac{T(r,fog)}{T(r,f)} = \lim\limits_{r\to\infty} \frac{T(r,fog)}{T(r,g)} = \infty$. Singh [8] proved some comparative growth properties of $\log T(r,fog)$ and $T(r,f)$. He [8] also raised the question of investigating the comparative growth of $\log T(r, fog)$ and $T(r, g)$ which he was unable to solve. However Lahiri [5] proved some comparative growth properties of $\log T(r, fog)$ and $T(r, g)$. In the paper we further investigate the results of Singh [8] and Lahiri [5] and prove some theorems on the comparative growths of $\log^{[2]} T(r\, fog)$ relative to $\log^{[2]} T(r, f^{(k)})$ and $\log^{[2]} T(r, g^{(k)})$ where $k = 0, 1, 2, 3, \ldots$ under different conditions. We also study the growths of $\log^{[2]} T(r\, fog)$ with $\log^{[2]} T(r, foh)$ for any three entire functions f, g and h. We do not explain the standard notations and definitions of the theory of entire and meromorphic functions because those are available in [9] and [4].

In the sequel we use the following notation :

$$\log^{[k]} x = \log(\log^{[k-1]}x) \text{ for } k = 1, 2, 3, \ldots \text{ and } \log^{[0]}x = x.$$

The following definitions are well known.

Definition 1. The order ρ_f and lower order λ_f of a meromorphic function f is defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2. The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f is defined as follows :

$$\overline{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log r}$$

If f is entire then

$$\overline{\rho}_f = \limsup_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r} \text{ and } \overline{\lambda}_f = \liminf_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 3. [6] Let f be a meromorphic function of order zero. Then ρ_f^*, λ_f^* and $\overline{\rho}_f^*$, $\overline{\lambda}_f^*$ are defined as follows :

$$\rho_f^* = \limsup_{r \to \infty} \frac{\log T(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and $$\overline{\rho}_f^* = \limsup_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r} \text{ and } \overline{\lambda}_f^* = \liminf_{r \to \infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}$$

If f is entire, then clearly

$$\rho_f^* = \limsup_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}, \quad \lambda_f^* = \liminf_{r \to \infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}$$

and $$\overline{\rho}_f^* = \limsup_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r}, \quad \overline{\lambda}_f^* = \liminf_{r \to \infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 4. The type σ_f of a meromorphic function f is defined as

$$\sigma_f = \limsup_{r \to \infty} \frac{T(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

If f is entire then

$$\sigma_f = \limsup_{r \to \infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma 1.[1] *If f is meromorphic and g is entire, for sufficiently large values of r,*

$$T(r, fog) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r, g), f).$$

Lemma 2. *Let f be meromorphic and g be transcendental entire such that* $\rho_f = 0$ and $\lambda_g < \infty$. Then $\rho_{fog} \leq \rho_f^* \rho_g$.

Proof. In view of Lemma 1 and inequality $T(r, g) \leq \log^+ M(r, g)$ we get

$$\begin{aligned}
\rho_{fog} &= \limsup_{r \to \infty} \frac{\log T(r, fog)}{\log r} \\
&\leq \limsup_{r \to \infty} \frac{\log T(M(r,g), f) + O(1)}{\log r} \\
&\leq \limsup_{r \to \infty} \frac{\log T(M(r,g), f)}{\log^{[2]} M(r,g)} . \limsup_{r \to \infty} \frac{\log^{[2]} M(r,g)}{\log r} \\
&= \rho_f^* \rho_g.
\end{aligned}$$

This proves the lemma.

3. Theorems. *In this section we present the main results of the paper.*

Theorem 1. *Let f and g be two entire functions such that (i) $\lambda_f > 0$, (ii) $\overline{\rho}_f < \infty$ and (iii) $0 < \lambda_g \leq \rho_g$.*

$$\text{Then } \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \geq \max\left\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\right\} \textit{ for } k = 0, 1, 2, 3, \ldots$$

Proof. We know that for $r > 0$([7])

$$T(r, fog) \geq \frac{1}{3} \log M\{\frac{1}{8} M(\frac{r}{4}, g) + o(1), f\} \qquad \ldots (1)$$

Since λ_f and λ_g are the lower orders of f and g respectively, for given ε and for all large values of r we get $M(r, f) > r^{\lambda_f - \varepsilon}$ and $\log M(r, g) > r^{\lambda_g - \varepsilon}$.

where $0 < \varepsilon < \min \{\lambda_f, \lambda_g\}$.

So from (1) we get for all large values of r,

$$T(r, fog) \geq \frac{1}{3} \{ \frac{1}{8} M(\frac{r}{4}, g) + o(1)\}^{\lambda_f - \varepsilon}$$

i.e., $$T(r, fog) \geq \frac{1}{3} \{ \frac{1}{9} M(\frac{r}{4}, g)\}^{\lambda_f - \varepsilon}$$

i.e., $\log T(r, fog) \geq O(1) + (\lambda_f - \varepsilon)\ (r/4)^{\lambda_g - \varepsilon}$

i.e., $\log^{[2]} T(r, fog) \geq O(1) + (\lambda_g - \varepsilon) \log r$ ... (2)

Also for a sequence of values of r tending to infinity,

$\log^{[2]} T(r, f^{(k)}) < (\overline{\lambda}_f + \varepsilon) \log r$ ...(3)

So from (2) and (3) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \geq \frac{O(1) + (\lambda_g - \varepsilon) \log r}{(\overline{\lambda}_f + \varepsilon) \log r} \quad \text{... (4)}$$

Since $\varepsilon(> 0)$ is arbitrary, we obtain from (4),

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \geq \frac{\lambda_g}{\overline{\lambda}_f} \quad \text{... (5)}$$

Again from (1) we get for a sequence of values of r tending to infinity,

$$\log T(r, fog) \geq O(1) + (\lambda_f - \varepsilon) \left(\frac{r}{4}\right)^{\rho_g - \varepsilon}$$

i.e., $\log^{[2]} T(r, fog) \geq O(1) + (\rho_g - \varepsilon) \log r$ ... (6)

Also for all large values of r we get

$\log^{[2]} T(r, f^{(k)}) < (\overline{\rho}_f + \varepsilon) \log r$ ...(7)

So from (6) and (7) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} > \frac{O(1) + (\rho_g - \varepsilon) \log r}{(\overline{\rho}_f + \varepsilon) \log r} \quad \text{... (8)}$$

As $\varepsilon(0 < \varepsilon < \rho_g)$ is arbitrary, we obtain from (8).

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \geq \frac{\rho_g}{\overline{\rho}_f} \quad \text{... (9)}$$

From (5) and (9) we get,

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \geq \max \left\{ \frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f} \right\}.$$

This proves the theorem.

Theorem 2. *Let f be meromorphic and g be entire satisfying (i)* $0 < \overline{\lambda}_f \leq \overline{\rho}_f$, *(ii)* $\rho_g < \infty$ *and (iii)* $\rho_f < \infty$.

Then $\displaystyle \liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \leq \min \left\{ \frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f} \right\}$ *for* $k = 0, 1, 2, 3, \ldots$

Proof. By Lemma 1 and the inequality $T(r, g) \leq \log^+ M(r, g)$ we obtain for all sufficiently large values of r,

$T(r, fog) \leq \{1 + o(1)\}\ T(M(r, g), f)$

i.e., $\log T(r, fog) \leq O(1) + (\rho_f + \varepsilon) \log M(r, g)$ ... (10)

Also for a sequence of values of r tending to infinity,

$$\log M(r, g) < r^{\lambda_g + \varepsilon} \quad \text{... (11)}$$

So from (10) and (11) it follows for a sequence of values of r tending to infinity,

$\log T(r, fog) < O(1) + (\rho_f + \varepsilon)\, r^{\lambda_g + \varepsilon}$

i.e., $\log^{[2]} T(r, fog) < O(1) + (\lambda_g + \varepsilon) \log r$ ... (12)

For all large values of r,

$\log^{[2]} T(r, f^{(k)}) > (\overline{\lambda}_f - \varepsilon) \log r$ ...(13)

where $0 < \varepsilon < \overline{\lambda}_f$.

Now combining (12) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} < \frac{O(1)+(\lambda_g+\varepsilon)\log r}{(\overline{\lambda}_f-\varepsilon)\log r}$$

i.e., $\liminf\limits_{r\to\infty} \dfrac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} < \dfrac{\lambda_g+\varepsilon}{\overline{\lambda}_f-\varepsilon}$.

As $\varepsilon\ (>0)$ is arbitrary it follows from above,

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} \le \frac{\lambda_g}{\overline{\lambda}_f} \qquad \text{... (14)}$$

Again for all sufficiently large values of r we obtain from (10),

$$\log^{[2]} T(r, fog) \le O(1) + (\rho_g + \varepsilon) \log r \qquad \text{...(15)}$$

Also for a sequence of values of r tending to infinity and for $0 < \varepsilon < \overline{\rho}_f$,

$$\log^{[2]} T(r, f^{(k)}) > (\overline{\rho}_f - \varepsilon) \log r \qquad \text{... (16)}$$

So from (15) and (16) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} < \frac{O(1)+(\rho_g+\varepsilon)\log r}{(\overline{\rho}_f-\varepsilon)\log r} \qquad \text{... (17)}$$

Since $\varepsilon(>0)$ is arbitrary we obtain from (17),

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} \le \frac{\rho_g}{\overline{\rho}_f} \qquad \text{... (18)}$$

Combining (14) and (18) we obtain,

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r,fog)}{\log^{[2]} T(r,f^{(k)})} \le \min\left\{\frac{\lambda_g}{\overline{\lambda}_f}, \frac{\rho_g}{\overline{\rho}_f}\right\}.$$

Thus the theorem is established.

The following theorem is natural consequence of Theorem 1 and Theorem 2.

Theorem 3. *If f and g be two entire functions with (i) $0 < \overline{\lambda}_f \le \overline{\rho}_f < \infty$, (ii) $\rho_f < \infty$ and (iii) $0 < \lambda_g \le \rho_g < \infty$.*

Then $$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})} \le \min\left\{\frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f}\right\}$$

$$\le \max\left\{\frac{\lambda_g}{\bar{\lambda}_f}, \frac{\rho_g}{\bar{\rho}_f}\right\} \le \limsup_{r\to\infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, f^{(k)})}$$

where $k = 0, 1, 2, 3, \ldots$

Theorem 4. *Let f be meromorphic and g be entire with (i) $0 < \rho_g < \infty$, (ii) $\sigma_g > 0$, (iii) $0 < \rho_{fog} < \infty$, (iv) $\sigma_{fog} < \infty$ and (v) $\rho_f^* < 1$.*

Then $$\liminf_{r\to\infty} \frac{T(r, fog)}{T(r, g)} = 0.$$

Proof. From the definition of type we have for arbitrary positive ε and for all large values of r,

$$T(r, fog) < (\sigma_{fog} + \varepsilon) r^{\rho_{fog}} \qquad \ldots (19)$$

Also for a sequence of values of r tending to infinity,

$$T(r, g) > (\sigma_g - \varepsilon) r^{\rho_g} \qquad \ldots (20)$$

In view of Lemma 2 from (19) and (20) we obtain for a sequence of values of r tending to infinity,

$$\frac{T(r, fog)}{T(r, g)} < \frac{(\sigma_{fog} + \varepsilon) r^{\rho_f^* \rho_g}}{(\sigma_g - \varepsilon) r^{\rho_g}} = \frac{\sigma_{fog} + \varepsilon}{\sigma_g - \varepsilon}\, r^{(\rho_f^* - 1)\rho_g} \qquad \ldots (21)$$

As $\varepsilon(> 0)$ is arbitrary, in view of condition (v) we obtain from (21)

$$\liminf_{r\to\infty} \frac{T(r, fog)}{T(r, g)} = 0.$$

This proves the theorem.

Remark 1. The condition $\rho_f^* < 1$ in Theorem 4 is essential which is evident from the following example.

Example 1. Let $f = z$ and $g = e^z$.

So $\rho_g = \sigma_g = 1 = \rho_{fog} = \sigma_{fog}$, $\rho_f = 0$ and $\rho_f^* = \infty$.

Now $T(r, fog) = T(r, g) = \dfrac{r}{\pi}$.

Then $\liminf\limits_{r \to \infty} \dfrac{T(r, fog)}{T(r, g)} = 1$.

Theorem 5. *If f, g and h be three entire functions such (i)* $0 < \lambda_f \le \rho_f < \infty$, *(ii)* $0 < \lambda_g \le \rho_g < \infty$ and *(iii)* $0 < \lambda_h < \infty$.

Then $\dfrac{\lambda_g}{\lambda_h} \le \liminf\limits_{r \to \infty} \dfrac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \le \limsup\limits_{r \to \infty} \dfrac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \le \dfrac{\rho_g}{\lambda_h}$.

Proof. In view of (12) we get for a sequence of values of r tending to infinity,

$$\log^{[2]} T(r, foh) \le O(1) + (\lambda_h + \varepsilon) \log r. \qquad \dots (22)$$

Now from (2) and (22) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \ge \frac{O(1) + (\lambda_g - \varepsilon)\log r}{O(1) + (\lambda_h + \varepsilon)\log r}$$

i.e., $\liminf\limits_{r \to \infty} \dfrac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \ge \dfrac{\lambda_g - \varepsilon}{\lambda_h + \varepsilon}$.

Since $\varepsilon(> 0)$ is arbitrary we obtain from above,

$$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \ge \frac{\lambda_g}{\lambda_h} \qquad \dots (23)$$

In view of (2) we get for all large values of r,

$$\log^{[2]} T(r, foh) \ge O(1) + (\lambda_h - \varepsilon) \log r. \qquad \dots (24)$$

Now from (15) and (24) it follows for all sufficiently large values of r,

$$\frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \le \frac{O(1) + (\rho_g + \varepsilon)\log r}{O(1) + (\lambda_h - \varepsilon)\log r}.$$

As $\varepsilon(> 0)$ is arbitrary we obtain,

$$\limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} \le \frac{\rho_g}{\lambda_h} \qquad \dots (25)$$

Thus the theorem follows from (23) and (25).

Remark 2. The inequality in Theorem 5 is best possible in the sense that '≤' cannot be replaced by '<' only as we see from the following example.

Example 2. Let $f = g = h = e^z$.

Then $\lambda_f = \rho_f = \lambda_g = \rho_g = \lambda_h = \rho_h = 1$.

Now $T(r, fog) = T(r, foh) \sim e^r / (2\pi^3 r)^{1/2}$.

So $$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} = \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} = 1$$

and $$\frac{\lambda_g}{\lambda_h} = \frac{\rho_g}{\lambda_h} = 1.$$

Now the following corollary immediately follows from Theorem 5.

Corollary 1. In addition to the conditions of Theorem 5 if g is of regular growth i.e., $\lambda_g = \rho_g$, then

$$\lim_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r, foh)} = \frac{\lambda_g}{\lambda_h} = \frac{\rho_g}{\lambda_h}.$$

The next theorem can be carried out in the line of Theorem 3.9 and Theorem 3.10 of [3] and so the proof is omitted.

Theorem 6. Let f be meromorphic and g be entire such that $0 < \overline{\lambda}_{fog} \le \overline{\rho}_{fog} < \infty$ and $0 < \overline{\lambda}_g \le \overline{\rho}_g < \infty$.

Then $$\liminf_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, g^{(k)})} \le \min \left\{ \frac{\overline{\lambda}_{fog}}{A\,\overline{\lambda}_g}, \frac{\overline{\rho}_{fog}}{A\,\overline{\rho}_g} \right\}$$

$$\le \max \left\{ \frac{\overline{\lambda}_{fog}}{A\,\overline{\lambda}_g}, \frac{\overline{\rho}_{fog}}{A\,\overline{\rho}_g} \right\} \le \limsup_{r \to \infty} \frac{\log^{[2]} T(r, fog)}{\log^{[2]} T(r^A, g^{(k)})}$$

for $k = 0, 1, 2, 3, \ldots$.

REFERENCES

1. W. Bergweiler, On the Nevanlinna Characteristic of a composite function, Complex Variables **10** (1988), pp. 225-236.

2. J. Clunie, The composition of entire and meromorphic functions, Mathematical essays dedicated to A. J. Macintyre, Ohio University Press (1970), pp. 75-92.

3. S. K. Datta, Further growth properties of composite entire and meromorphic functions, Jour. Pure Math. **19** (2002), pp. 19-29.

4. W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).

5. I. Lahiri, Growth of composite integral functions, Indian J. Pure Appl. Math., **20** (9), September (1989), pp. 899-907.

6. L. Liao and C. C. Yang, On the growth of composite entire functions, Yokohama Math. J. Vol. **46** (1999), pp. 97-107.

7. K. Niino and C. C. Yang, Some growth relationships on factors of two composite entire functions, Factorization Theory of Meromorphic. Functions and Related Topics, Marcel Dekker, Inc. (New York and Basel), (1982), pp. 95-99.

8. A. P. Singh, Growth of composite entire functions, Kodai Math. J. **8** (1985), pp. 99-102.

9. G. Valiron, Lectures on the General Theory of Integral Functions, Chelsea Publishing Company (1949).

Department of Mathematics
Krishnath College
Berhampore, Murshidabad
West Bengal-742101.

Address for communication :
25, School Road, Kalianibash
Barrackpore, P .O.-Nonachandanpukur,
Dist.- 24 Parganas (North)
P .S.- Titagarh,
West Bengal, India
Pin -743102.

Jour. Pure Math., Vol. 21, 2004, pp. 49-54

RELATION BETWEEN FUZZY BILINER FORMS AND BOUNDED LINEAR OPERATORS

N. R. DAS & R. K. MISRA

ABSTRACT. In this paper we have introduced fuzzy bounded bilinear form on fuzzy normed linear spaces. Relations between fuzzy bounded bilinear forms and bounded linear operators are also established.

Key Words. Fuzzy normed linear space, fuzzy bilinear form, bounded linear operator, fuzzy bounded bilinear form.

1. PRELIMINARIES.

In this section, we recall some basic definitions and results useful in the sequel. For details, we refer to [2],[4],[5],[6]

Definition 1.1. [6] A fuzzy real number η is a fuzzy set on IR, the set of real numbers, i.e. a mapping $\eta : IR \to I = [0, 1]$ associating with each real number t its grade of membership $\eta(t)$.

Definition 1.2. [6] A fuzzy real number η is called convex if $\eta(t) \geq \eta(s) \wedge \eta(r) = \min(\eta(s), \eta(r))$ where $s \leq t \leq r$.

Definition 1.3. [6] A fuzzy real number η is called normal if there exists $t_0 \in IR$ such that $\eta(t_0) = 1$.

Definition 1.4. [6] The α-level set of a fuzzy real number η denoted by $[\eta]_\alpha$, $0 \angle \alpha \leq 1$ is defined as $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$.

Definition 1.5. [6] A fuzzy real number η is called upper semi-continuous if for all $\in > 0$, η^{-1} ([0, $a + \in$]); for all $a \in I$ is open in the usual topology of IR.

Remark 1.6. [6] It can be easily seen that the α-level set $[\eta]_\alpha$ of an upper semi-continuous, normal, convex, fuzzy real number η, for each α, $0 < \alpha \leq 1$ is a closed interval $[a^\alpha, b^\alpha]$, where $a^\alpha = -\infty$ and $b^\alpha = \infty$ are also admissible. The set of all upper semi-continuous, normal, convex, fuzzy real numbers is denoted by $IR(I)$.

Since each $r \in IR$ can be considered as a fuzzy real $\bar{r}$ defmed by $\bar{r}(t) = 1$ if $t = r$ and $\bar{r}(t) = 0$ if $t \neq r$. IR can be embedded in $IR(I)$

Definition 1.7. [6] A fuzzy real number η is called non-negative if $\eta(t) = 0$ for all $t < 0$. The set of all non-negative fuzzy real numbers is denoted by $IR^*(I)$

Definition 1.8. [5] The arithmetic operations +, – and / on $IR(I) \times IR(I)$ are defined by

(i) $(\eta_1 \oplus \eta_2)(t) = \sup_{s \in IR} \min(\eta_1(s), \eta_2(t - s)), t \in IR$

(ii) $(\eta_1 \ominus \eta_2)(t) = \sup_{s \in IR} \min(\eta_1(s), \eta_2(s - t)), t \in IR$

(iii) $(\eta_1 \ominus \eta_2)(t) = \sup_{\substack{s \in IR \\ s \neq O}} \min((\eta_1(s), \eta_2(t / s)), t \in IR$

(iv) $(\eta_1 / \eta_2)(t) = \sup_{s \in IR} \min(\eta_1(ts), \eta_2(s)), t \in IR$

The additive and multiplicative identities in $IR(I)$ are $\bar{0}$ *and* $\bar{1}$ respectively where $\bar{0}(t) = 1$ if $t = 0$ and $\bar{0}(t) = 0$ if $t \neq 0$; $\bar{1}(t) = 1$ if $t = 1$ and $\bar{1}(t) = 0$ if $t \neq 1$.

Definition 1.9. [6] Let η be a fuzzy real number. The absolute value $|\eta|$ of the fuzzy real number η is defined by $|\eta|(t) = \max(\eta(t), \eta(-t))$ if $-t \geq 0$ and $|\eta|(t) = 0$ if $t < 0$.

Definition 1.10. [4] Let X be a linear space over the field IR. Let $\|.\| : X \to IR^*(I)$ be a mapping and let the mappings $L, R : [0, 1] \times [0, 1] \to [0, 1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$.

Write $[\|x\|]_\alpha = \left[|||x|||_1^\alpha, |||x|||_2^\alpha\right]$ for $x \in X$, $\alpha \in (0, 1]$ and suppose for all $x \in X$, $x \neq 0$, there exists $\alpha_0 \in (0, 1]$ (independent of x) such that for all $\alpha \leq \alpha_0$

(A) $|||x|||_2^\alpha < \infty$

(B) $\inf |||x|||_1^\alpha > 0$

The quadruple $(X, \|.\|, L, R)$ is called a fuzzy normed linear space and $\|.\|$ a fuzzy norm, if

(i) $\|x\| = \bar{0}$ if and only if $x = 0$

(ii) $\|rx\| = |r| \|x\|$, $x \in X$, $r \in IR$

(iii) for all $x, y \in X$

(a) Whenever $s \leq |||x|||_1^1$, $t \leq |||y|||_1^1$, and $s + t \leq |||x + y|||_1^1$, $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$

(b) Whenever $s \geq |||x|||_1^1$, $t \geq |||y|||_1^1$, and $s + t \geq |||x + y|||_1^1$, $\|x + y\|(s + t) \leq L(\|x\|(s), \|y\|(t))$

Remark 1.11. In the sequel, we take $L(x, y) = \min(x, y)$ and $R(x, y) = \max(x, y)$ for $x, y \in (0, 1]$, in this case we write $(X, \|\cdot\|)$, or simply X for $(X, \|\cdot\|, L, R)$.

Proposition 1.11. [4] In a fuzzy normed linear space $(X, \|\cdot\|, \min, \max)$ the triangular inequality (iii) of definition 1.10 is equivalent to $\| x + y \| \leq \| x \| + \| y \|$

Definition 1.13. [2] Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be two fuzzy normed linear spaces and $F : X \to Y$ be a linear map from X into Y. Then F is said to be bounded in a fuzzy subset A of X if for some $M > 0$, $\| F(x) \|_2 \leq \overline{M}$ for all $x \in X$ with $A(x) > 0$

Proposition 1.14. [2] Let $(X, \|\cdot\|_1)$ and $(Y, \|\cdot\|_2)$ be two fuzzy normed linear spaces and $F : X \to Y$ be a linear map. The following conditions are equivalent :

(i) F is bounded in $\overline{S}_\alpha(r, 0)(x)$ for some $r > 0$, $\alpha \in (0, 1]$, where $\overline{S}_\alpha(r, 0)$ is a fuzzy subset of X defined by $\overline{S}_\alpha(r, 0)(x) = \alpha$ if and only if $|||x|||_2^\alpha \leq r$ and $\overline{S}_\alpha(r, 0)(x) = 0$ otherwise.

(ii) F is continuous at 0.

(iii) F is continuous on X.

(iv) F is uniformly continuous on X.

(v) $\| F(x) \| \leq \overline{\beta} \ominus \| x \|$ for all $x \in X$ and some $\beta > 0$

Definition 1.15 [2] Let X, Y be two vector spaces over the field of reals IR . A mapping $B : X \times Y \to IR^*(I)$ is called a fuzzy bilinear form if and only if the following conditions hold :

For $x, y \neq 0$, $\alpha, \beta \in IR$,

(i) $B(\alpha x_1 + \beta x_2, y) = \alpha B(x_1, y) \oplus \beta B(x_2, y)$

(ii) $B(x, \alpha y_1 + \beta y_2) = \alpha B(x, y_1) \oplus \beta B(x, y_2)$ and

(iii) $B(x, y) = 0$ if at least one of x and y is zero.

Definition 1.16. [2] The two vector spaces X and Y over the field of reals IR are said to form a duel pair $< X, Y >$ under the fuzzy bilinear form $B(x, y)$ if the following conditions hold :

(i) $B : X \times Y \to IR^*(I)$ is a fuzzy bilinear form.

(ii) If for some $x \in X$, $B(x, y) = 0$ for all $y \in Y$, then $x = 0$.

(iii) If for some $y \in Y$, $B(x, y) = 0$ for all $x \in X$, then $y = 0$.

2. MAIN RESULTS.

Definition 2.1. Let X and Y be two fuzzy normed linear spaces and $B : X \times Y \to IR^*(I)$ be a fuzzy bilinear form. Then B is said to be fuzzy bounded bilinear form if $| B(u,v) | \leq \overline{d} \ominus \| u \| \ominus \| v \|$ for some $d > 0$, $u \in X$, $v \in Y$.

Proposition 2.2. Let X and Y be two fuzzy normed linear spaces. Let $p : Y \to IR^*(I)$ defined by $P(Av) = \text{Sup} \mid B(u, Av) \mid$, $u, v \in X$, where $A : X \to Y$ is a bounded linear operator and $B : X \times Y \to IR^*(I)$ is a bilinear from where X and Y form a duel pair. Then p is a fuzzy norm on Y.

Proof. For all $u, v \in X$, $u \neq 0$, $v \neq 0$, there exists $\alpha_0 \in (0, 1]$ independent of u, v such that for all $\alpha \leq \alpha_0$

(A) $|||p(Av)|||_2^{\alpha} < \infty$

(B) inf $|||p(Av)|||_1^{\alpha} > 0$

For, $u\ v \in X$

(i) $p\ (Av) = \overline{O} \Rightarrow \underset{\|u\|(1)=1}{\text{Sup}} \mid B\ (u, Av) \mid = 0$

$\Rightarrow [| B(u, Av) |]_1^{\alpha} = 0\ \&\ [| B(u, Av) |]_2^{\alpha} = 0$

$\Rightarrow B\ (u, Av) = \overline{O}$ for all u such that $\|u\|(1) = 1$

$\Rightarrow Av = 0$ since $u \neq 0$

Also, $Av = 0 \Rightarrow B\ (u, Av) = \overline{O}$ for all u such that $\|u\|(1) = 1$

$\Rightarrow [| B(u, Av) |]_1^{\alpha} = 0$ and $[| B(u, Av) |]_2^{\alpha} = 0$

$\Rightarrow \underset{\|u\|(1)=1}{\text{Sup}} \mid B\ (u, Av) \mid = \overline{O}$

$\Rightarrow p(Av) = \overline{O}$

Thus, $p\ (Av) = \overline{O}$ if and only if $Av = 0$

(ii) $p\ (r\, Av) = \underset{\|u\|(1)=1}{\text{Sup}} \mid B\ (u, rAv) \mid$, $r \in IR$

$= \underset{\|u\|(1)=1}{\text{Sup}} \mid rB\ (u, Av) \mid$

$= \underset{\|u\|(1)=1}{\text{Sup}} \mid r \| B(u, Av)1$

$= \mid r \mid \underset{\|u\|(1)=1}{\text{Sup}} \mid B\ (u, Av) \mid$

$= \mid r \mid p(Av)$

(iii) for, $u, v_1, v_2 \in X$

$P\ (A\ v_1 + A\ v_2) = \underset{\|u\|(1)=1}{\text{Sup}} \mid B\ (u, A\ V_1 + A\ v_2) \mid$

$$= \operatorname*{Sup}_{\| u \|(1) = 1} | B(u, A v_1) \oplus B(u, A v_2) |$$

$$\leq \operatorname*{Sup}_{\| u \|(1) = 1} | B(u, Av_1) | \oplus \operatorname*{Sup}_{\| u \|(1) = 1} | B(u, Av_2) |$$

$$= P(A v_1) \oplus P(A v_2)$$

Thus, p is a fuzzy norm on Y.

Proposition 2.3. Let X and Y be two fuzzy normed linear spaces and let $B : X : \times Y \to \mathbb{R}^*(I)$ be a fuzzy bilinear form, $< X, Y >$ be a dual pair. Then there exists a one to one correspondences between fuzzy bounded bilinear form a : $X \times X \to \mathbb{R}^*(I)$ and the linear operator $A : X \to Y$ which is bounded with respect to fuzzy norm defined by $\|Av\| = \operatorname*{Sup}_{\|u\|(1)=1} |B(u, Av)|$

Proof. Let $A : X \to Y$ be linear and bounded,

We set $a(u, v) = B(u, Av)$

Now for some constant $M > 0$, $N > 0$

$$| a(u,v) | = | B(u, Av) | \leq \overline{M} \odot \| u \| \odot \| Av \|$$
$$\leq \overline{M} \odot \| u \| \odot \overline{N} \odot \| v \|$$
$$= \overline{M} \odot \overline{N} \odot \| u \| \odot \| v \| \text{ since } A \text{ is bounded.}$$

This implies that a is bounded.

Also for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$, $u, v \neq 0$

(i) $a(\alpha u_1 + \beta u_2, v) = B(\alpha u_1 + \beta u_2, Av)$
$= \alpha B(u_1, Av) \oplus \beta B(u_2, Av)$
$= \alpha a(u_1, v) \oplus \beta a(u_2, v)$

(ii) $a(u, \gamma v_1 + \delta v_2) = B(u, A(\gamma v_1 + \delta v_2))$
$= B(u, \gamma A v_1 + \delta A v_2)$
$= \gamma B(u, Av_1) \oplus \delta B(u, Av_2)$
$= \gamma a(u, v_1) \oplus \delta a(u, v_2)$

(iii) Since $a(u, v) = B(u, Av)$ and B is a bilinear form so
$a(u, v) = \overline{O}$ if and only if $B(u, Av) = \overline{O}$

That is, $a(u, v) = \overline{O}$ if at least one of u and v is zero.

Thus a is a fuzzy bilinear form which is bounded also.

Conversely, let $a : X \times X \to \mathbb{R}^*(I)$ is fuzzy bilinear and bounded.

Set $a(u, v) = B(u, Av)$

We need to show that $A : X \to Y$ is a bounded linear map.

Now,

$$a(u,\ \alpha v_1 + \beta\ v_2) = \alpha\ a(u,\ v_1) \oplus \beta\ a(u,\ v_2)$$

$$\Rightarrow\ B\ (u,\ A\ (\alpha v_1 + \beta\ V_2)) = \alpha\ B\ (u,\ Av_1) \oplus \beta\ B\ (u,\ Av_2) = B\ (u,\ \alpha Av_1 + \beta\ Av_2)$$

$$\Rightarrow\ A\ (\alpha v_1 + \beta\ v_2) = \alpha\ Av_1 + \beta\ Av_2$$

This implies that A is linear.

Also, $|\ B\ (u,\ Av)\ | = |\ a\ (u,\ v)\ | \leq \bar{k} \odot \|\ u\ \| \odot \|\ v\ \|$ for some $k > 0$.

$$\Rightarrow\ \text{Sup}\ |\ B\ (u,\ Av)\ | \leq \bar{k} \odot \|\ v\ \|$$

$$\|\ u\ \|(1) = 1$$

$$\Rightarrow\ \|\ Av\ \| \leq \bar{k} \odot \|\ v\ \|$$

$\Rightarrow$ A is bounded.

Thus, A is linear and bounded.

CONCLUSION

We have Shown that it is possible to establish a relation between fuzzy bounded bilinear forms, with bounded linear operators on fuzzy normed linear spaces. With continued effort on research in fuzzy bilinear form we would enrich the theory of bilinear form.

REFERENCES

1. N. R. Das, Pankaja Das and Ajanta Choudhury, Absolute value like fuzzy real number and fuzzy real valued sequence space, Jour. Of Fuzzy Math. 4 (2), (1996), 421-433.
2. Pankaja Das, A Study of Some Aspect of Fuzzy Normed Linear Spaces, Ph. D. Thesis, Gauhati University (1999).
3. N. R. Das and Pankaja Das : Fuzzy topology generated by fuzzy norm, Fuzzy Sets and System, 107 (1999) 349-354.
4. C. Felbin : The completion of a fuzzy normed linear space, J. Math. Anal. Appl.,174 (1993) no. 2, 428-440.
5. M. M. Gupta and A. Kaufman : Introduction to Fuzzy Arithmetic-Theory and Applications; Van Nostrand Reinhold company, New York.
6. O. Kaleeva and S. Seikkla : On fuzzy metric spaces, Fuzzy sets and systems, 12 (1984), 215-229.
7. Eberhard Zeidler, Non Linear Functional Analysis and its Applications II/A. Linear Monotone Operators. Springer-Verlag, New York.

Department of Mathematics
Gauhati University
Guwahati -781014
Assam, India

Department of Mathematics
Darrang College
Tezpur -784001
Assam, India.

Jour. Pure Math., Vol. 21, 2004, pp. 55-64

SOME FIXED POINT THEOREMS IN FUZZY METRIC SPACES

SIKHA SAIKIA (BORDOLOYE)

ABSTRACT : In this paper, two fixed point theorems, one in complete fuzzy metric space and other in fuzzy metric space have been proved in fuzzy structure. Boyd and Wong (for the lst theorem), Sehgal (for the 2nd theorem) had obtained these two fixed-point theorems in the classical structure of functional analysis, The transformation has been done in the sense of Kaleva and Seikkala.

Key words : Fuzzy metric space, complete fuzzy metric space.

AMS CLASSIFICATION 1991 : 47H10,54H25

1. INTRODUCTION

In 1984, Kaleva O. and Seikkala S[9] generalized the notion of the metric space by setting the distance between two points to be a non-negative number which seemed to be a more natural way to define the fuzzy metric space. They defined ordering and triangular inequality and also developed the notion of fuzzy metric space, its properties and gave some examples. They proved some fixed point theorems in fuzzy metric spaces. Adhering to the step of Kaleva and Seikkala, many researchers have been doing works on fuzzy metric spaces and fuzzy normed linear spaces. The works of Sharma[11], Das and Basu[4], Badard[1], Bose and Sahani[2], C. Felbin[5] and Jung, Cho and Kim[8] are worth mentioning. Here, we have proved two fixed point theorems which are fuzzy annlogue of fixed point theorems obtained by Boyd and Wong and by Sehgal in metric spaces of classical structure.

2. PRELIMINARIES

A fuzzy number is a fuzzy subset of real numbers i.e. a mapping $x : \mathrm{R} \to [0, 1]$ associating each real number t with its grade of membership $x(t)$.

A fuzzy number x is convex if $x(t) \geq \min\ (x(s), x(r))$ where $s \leq t \leq r$.

Definition 2.1 For a α with $0 < \alpha \leq 1$, the α level set of a fuzzy nwnber x as $[x]_\alpha$ is defined by $[x]_\alpha = \{t : x(t) \geq \alpha\}$.

A fuzzy number is convex iff its α-level sets are convex sets in R.

A fuzzy number x is called normal if there exists a $t_0 \in \mathbf{R}$ such that $x(t_0) = 1$. The set of all upper semi continuous convex normal fuzzy numbers are denoted by E. For any $x \in E$, the α-level set is defined by

$$[x]_\alpha = [a^\alpha, b^\alpha],\ 0 < \alpha \leq 1$$

The values $a^\alpha = -\infty$ and $b^\alpha = \infty$ are admissible.

Each $x \in \mathbf{R}$ can be represented by the fuzzy number $\bar{x}$ defined by $\bar{x}(t) = 1$ if $t = x$

$\bar{x}(t) = 0$ if $t = x$

then $\bar{x} \in E$.

Deffinition 2.2 A fuzzy number x is called non-negative if $x(t) = 0$ for all $t < 0$. G denotes the set of all non-negative fuzzy numbers of E.

Lemma 2.1. [9] *Let $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$, be a given family of non empty intervals.*

If (*a*) $[a^{\alpha_1}, b^{\alpha_1}] \supset [a^{\alpha_2}, b^{\alpha_2}]$ *for all* $0 < \alpha_1 \leq \alpha_2$

and (*b*) $[\lim_{k \to \infty} a^{\alpha_k}, \lim_{k \to \infty} b^{\alpha_k}] = [a^\alpha, b^\alpha]$

Whenever (α_k) is an increasing sequence in $(0, 1]$ converging to α, then the family $[a^\alpha, b^\alpha]$ represents the α-level sets of a fuzzy number x in E.

Conversely, if $[a^\alpha, b^\alpha]$, $0 < \alpha \leq 1$ are the α-level sets of a fuzzy number $x \in E$, then the conditions (a) and (b) are satisfied.

Definition 2.3 [9] Let X be a non empty set, d be a mapping from $X \times X$ into G and let the mappings $L, R : (0, 1] \times (0, 1] \to (0, 1]$ be symmetric, non decreasing in both arguments and satisfy $L(0, 0) = 0$ and $R(1, 1) = 1$.

Denote

$$[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)] \qquad \text{for } x, y \in X \text{ and } 0 < \alpha \leq 1.$$

The quadruple (X, d, L, R) is called a fuzzy metric space and d a fuzzy metric if

(i) $d(x, y) = \bar{0}$ iff $x = y$

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

(iii) For all $x, y, z \in X$

(1) $d(x, y)(s + t)) \geq L(d(x, z)(s), d(z, y)(t))$ whenever

$s \leq \lambda_1(x, z)$, $t \leq \lambda_1(z, y)$ and $s + t \leq \lambda_1(x, y)$

(2) $d(x, y)(s + t) \leq R(d(x, z)(s), d(z, y)(t))$ whenever

$s \geq \lambda_1(x, z)$, $t \geq \lambda_1(z, y)$ and $s + t \geq \lambda_1(x, y)$

Lemma 2.2. and lemma 2.3. [9] *The triangle inequalities (iii)(2) and (iii)(1) with R =* Max *and L =* Min *respectively are equivalent to the triangle inequalities*

$$\rho_\alpha(x, y) \leq \rho_\alpha(x, z) + \rho_\alpha(z, y) \text{ for all } \alpha \in (0, 1] \text{ and } x, y, z \in X$$

and

$$\lambda_\alpha(x, y) \leq \lambda_\alpha(x, z) + \lambda_\alpha(z, y) \text{ for all } \alpha \in (0, 1] \text{ and } x, y, z \in X$$

Definition 2.4 [9] Let (X, d, L, R) be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be converged to $x \in X$ denoted by $\lim_{n \to \infty} x_n = x$ if $\lim_{n \to \infty} a_n^\alpha = a^\alpha$ and $\lim_{n \to \infty} b_n^\alpha = b^\alpha$ for all $\alpha \in (0, 1]$ where $[x_n]_\alpha = [a_n^\alpha, b_n^\alpha]$ and $[x]_\alpha = [a^\alpha, b^\alpha]$.

Definition 2.5 [9] Let $(X, d, L, R]$ be a fuzzy metric space. A sequence $\{x_n\}$ is said to be converged to $x \in X$ denoted by $\lim_{n \to \infty} x_n = x$ if $\lim_{n \to \infty} d(x_n, x) = 0$.

Lemma 2.3. [5] A sequencc $\{x_n\}$ in a fuzzy metric space (X, d, L, R) converges to $x \in X$ iff $\lim_{n \to \infty} \rho_\alpha(x_n, x) = 0$ for all $\alpha \in (0, 1]$.

Defition 2.6 [9] A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence if $\lim_{\substack{m \to \infty \\ n \to \infty}} d(x_m, x_n) = \bar{0}$.

Lemma 2.4. [5] A sequence $\{x_n\}$ in a fuzzy metric space (X, d, L, R) is called a Cauchy sequence iff $\lim_{\substack{m \to \infty \\ n \to \infty}} \rho_\alpha(x_m, x_n) = 0$.

Defltion 2.7 [9] A fuzzy metric space X is called complete if each cauchy sequence in X converges in X.

Example 2.1. [9] Let $\{\lambda_\alpha\}_{0 < \alpha \leq 1}$ be two families of pseudometrics on a non empty set X such that for all $(x, y) \in X \times X$, $\lambda_1(x, y) \leq \rho_1(x, y)$, $\lambda_\alpha(x, y)$ is left continuous and non decreasing in α and $\rho_\alpha(x, y)$ is left continuous and non increasing in α. If furthermore, the family $\{\rho_\alpha\}$ is saturated, i.e. if $\rho_\alpha(x, y) = 0$ for all $\alpha \in (0, 1]$ then $x = y$, then the equation

$$[d(x, y)]_\alpha = [\lambda_\alpha(x, y), \rho_\alpha(x, y)] \text{ for all } \alpha \in (0, 1] \text{ and } x, y \in X,$$

defines a fuzzy metric $d : X \times X \to G$ which satisfies the triangle inequality

$$d(x, y) \leq d(x, z) + d(z, y) \text{ for all } x, y, z \in X.$$

That the equation defines a fuzzy number follows from lemma 2.1 [9]. If the family is given then we may choose, $\lambda_\alpha(x, y) = \alpha\, \rho_1(x, y)$.

Remark 2.1. [9] If the fuzzy distances $d(x, y)$ in fuzzy metric space (X, d, L, Max) satisfy $\lim_{t \to \infty} d(x, y)(t) = 0$ for all x, y $\in X$, then $\rho_\alpha(x, y) < \infty$ for all $\alpha \in (0, 1]$.

Proposition 2.1 [8] Let (X, d_i, L, Max) be two complete fuzzy metric spaces such that $\lim_{t \to \infty} (x, y)(t) = 0$ for all $x, y \in X_i$, $i = 1, 2$. Let $f : X_1 \to X_2$ be a closed continuous mapping and let $\phi : f(X_1) \to (-\infty, \infty]$ be a proper lower semi continuous function, bounded from below. Assume that for each $u \in X_1$ with $\inf_{x \in X_1} \phi(f(x)) \prec \phi(f(u))$, there exists $v \in X_1$ such that $u \neq v$ and

$$\max\{\rho_{1\alpha}(u, v), \rho_{2\alpha}(f(u), f(v))\} \leq \phi(f(u)) - \phi(f(v)) \text{ for all } \alpha \in (0, 1].$$

3. MAIN RESULTS

Here, we will prove two fixed point theorems which are fuzzy analogues of fixed point theorems done by Boyd and Wong (the first theorem) and Sahgal (the second theorem) in the classical analysis. Throughout the part, X denotes fuzzy metric space (X, d, L, Max) with $\lim_{n \to \infty} d(x, y)(t) = 0$, for all $x, y \in X$.

$\lim_{n \to \infty} d(x, y)(t) = 0$ follows that $\rho_\alpha(x, y) < \infty$ for all $\alpha \in (0, 1]$. We recall that $\rho_\alpha(x, y)$ is the right end point of the α-level interval of $d(x, y)$ and $\rho_\alpha(x, y)$ is non-increasing and left continuous in α.

Theorem 3.1. *Let T be a self-mapping on a complete fuzzy metric space* (X, d, L, Max). *Suppose, there exists a function* φ, *upper semi continuous from right from* R^+ *into itself such that*

$$\rho_\alpha(Tx, Ty) \leq \varphi(\rho_\alpha(x, y)) \textit{ for all } \alpha \in (0, 1] \textit{ and for all } x, y \in X.$$

If $\varphi(t) < t$ *for each* $t > 0$, *then* T *has a unique fixed point* ξ *in* X *and for every* x *in* X, $\lim_{n \to \infty} T^n x = \xi$,

Proof : Let for any $x \in X$, $x_n = T^n x$ So we have a sequence $\{T^n x\} = \{x_n\}$ in X.

Let $\rho_\alpha(x_n, x_{n+1}) = (\rho_n)_\alpha$ for all $\alpha \in (0, 1]$

So $(\rho_n)_\alpha = \rho_\alpha(x_n, x_{n+1}) = \rho_\alpha(T^n x, x^{n+1})$ for all $\alpha \in (0, 1]$.

Since d is a fuzzy metric on X, so we can assume that $(\rho_n)_\alpha > 0$ for all $\alpha \in (0, 1]$ and for $n \geq 0$.

Then for $n > 1$, we have

$(\rho_n)_\alpha = \rho_\alpha(Tx_{n-1}, Tx_n) \leq \varphi(\rho_\alpha(x_{n-1}, x_n)) = \varphi(\rho_{n-1})_\alpha < ((\rho_{n-1})_\alpha$ for all $\alpha \in (0, 1]$. Therefore $\{\rho_n\}$ is a convergent sequence. Let $[\lim\limits_{n\to\infty} \rho_n]_\alpha = \rho_\alpha$ for all $\alpha \in (0, 1]$.

Then $(\rho_n)_\alpha \leq \varphi[(\rho_{n-1})_\alpha]$

$\Rightarrow [\lim\limits_{n\to\infty} \rho_n]_\alpha \leq \varphi\ [\lim\limits_{n\to\infty} \rho_{n-1}]_\alpha$

$\Rightarrow (\rho)_\alpha \leq \varphi[(\rho)_\alpha]$ for all $\alpha \in (0, 1]$; which is a contradiction to $\varphi[(\rho)_\alpha] \dot{<} (\rho)_\alpha$ for $\rho > 0$. Therefore $(\rho)_\alpha = 0$ for all α level sets in $(0, 1]$.

Now, to prove the theorem we have to show that $\{T^n x\} = \{x_n\}$ is a Cauchy sequence. On the contrary, we assume that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\in > 0$ and for each positive integer z, there exists $p(z)$ and $q(z)$ with $z \leq p(z) < q(z)$ such that

$$\rho_\alpha(p(z), q(z)) \geq \in$$

Thus we can have many integers satisfying the above inequality. So, without loss of generality we can assume that $q(z)$ is the smallest of all those integers greater than $p(z)$ which satisfies the above inequality.

Let $\rho_\alpha(x_{p(z)}, x_{q(z)}) = r_z$ for all α level sets in $(0, 1]$.

Then we have

$\in\ \leq r_z \leq \rho_\alpha(x_{p(z)}, x_{q(z)-1}), + \rho_\alpha(x_{q(z)-1}, x_{q(z)})$ for all $\alpha \in (0, 1]$

$\leq \in + (\rho_{q(z)-1})_\alpha$ for all $\alpha \in (0, 1]$

This implies $\lim\limits_{z\to\infty} r_z = \in$

Also we have

$\in\ \leq r_z \leq \rho_\alpha(x_{p(z)}, x_{p(z)+1}) + \rho_\alpha(x_{p(z)+1}, x_{q(z)+1}) + \rho_\alpha(x_{q(z)+1}, x_{q(z)})$ for all $\alpha \in (0, 1]$

i.e. $\in\ \leq r_z \leq (\rho_{p(z)})_\alpha + \varphi(r_z) + (\rho_{q(z)})_\alpha$ for all $\alpha \in (0, 1]$

when $z \to \infty$, then $\in\ \leq \varphi(\in)$ for all $\alpha \in (0, 1]$ which is a contradiction. Therefore our assumption is wrong and hence the sequence $\{x_n\}$ or $\{T^n x\}$ is a Cauchy sequence in X. Since (X, d, L, Max) is complete fuzzy metric space, so, $\{T^n x\}$ must be a convergent sequence.

Let $\{T^n x\}$ converges to a point ξ (say) of X. That is, $\lim\limits_{n\to\infty} T^n x = \xi$.

Now, we have to show that ξ is a fixed point of T i.e. $T\xi = \xi$.

$$\rho_\alpha(T\xi, \xi) \leq \rho_\alpha(T\xi, T^n x) + \rho_\alpha(T^n x, \xi) \text{ for all } \alpha \in (0, 1]$$

or, $\rho_\alpha(T\xi, \xi) \leq \rho_\alpha(T\xi, Tx_{n-1}) + \rho_\alpha(T^n x, \xi)$ for all $\alpha \in (0, 1]$

$$\leq \varphi\{\rho_\alpha(\xi, x_{n-1})\} + \rho_\alpha(x_n, \xi) \text{ for all } \alpha \in (0, 1]$$

When $n \to \infty$, then the expression in the right hand side tends to 0.

Therefore, $\rho_\alpha(T\xi, \xi) = 0$

i.e. $T\xi = \xi$. Therefore ξ is a fixed point of T.

If possible, let there be another such fixed point η in X such that $T\eta = \eta$.

Now $\rho_\alpha(\xi, \eta) = \rho_\alpha(T\xi, T\eta) \leq \varphi\{\rho_\alpha(\xi, \eta)\}$ for all $\alpha \in (0, 1]$.

This implies, $\rho_\alpha(\xi, \eta) = 0$ for all $\alpha \in (0, 1]$.

Hence $T\xi = \xi$ is the unique fixed point of T. This completes the proof of the theorem.

Theorem 3.2. *Let $(X, d, L,$ Max$)$ be a fuzzy metric space and $T : X \to X$ be a continuous mapping such that for all x, y in X with $x \neq y$,* we have for all α-level sets in *$(0, 1]$.*

$$\rho_\alpha(Tx, Ty) \prec \max\{\rho_\alpha(x, Tx), \rho_\alpha(y, Ty), \rho_\alpha(x, y)\}$$

Supposing that for some z in X, the sequence $\{T^n z\}$ has a cluster point ξ. Then the sequence $\{T^n z\}$ converges to ξ of X and ξ is the unique fixed point of T.

Proof : Given that T is a continuous mapping in a fuzzy metric space $(X, d, L,$ Max$)$. Let for some non-negative integer n and for some $z \in X$, we have

$$T^n z = T^{n+1} z = T^{n+2} z = \ldots, \text{ and therefore } \lim_{n\to\infty} T^n z = \xi$$

$$\text{Or, } \lim_{n\to\infty} \rho_\alpha(T^n z, \xi) = 0 \text{ for all } \alpha \in (0, 1].$$

Then $T\xi = T \lim\limits_{n\to\infty} T^n z = \lim\limits_{n\to\infty} T\, T^n z = \lim\limits_{n\to\infty} T^{n+1} z = \xi$ and ξ is a fixed point of T. To prove that ξ is a unique fixed point, let there be another point η in X such that $\lim\limits_{n\to\infty} T^n z = \eta$ and $T\eta = \eta$.

$$\rho_\alpha(\xi, \eta) = \rho_\alpha(T\xi, T\eta) < \max\{\rho_\alpha(\xi, T\xi), \rho_\alpha(\eta, T\eta), \rho_\alpha(\xi, \eta)\}$$

$$= \max\{0, 0, \rho_\alpha(0, 1)\}$$

That implies, $\rho_\alpha(\xi, \eta) = 0$ for all $\alpha \in (0, 1]$

Therefore, $\xi = \eta$ and ξ is the unique fixed point of T.

Now, we will prove that for all non-negative integer n and $z \in X$, $\lim_{n\to\infty} T^n z = \xi$ and ξ is unique.

Assuming that for all non negative n, $\rho_\alpha(T^n z, T^{n+1}z) \succ 0$ for all $\alpha \in (0, 1]$.

Now, we define a function $U(y) = \rho_\alpha(y, Ty)$ for all $y \in X$ and for all $\alpha \in (0, 1]$. U is continuous because for any sequence $\{y_n\}$ in X converging to y of X, we have $Ty_n \to Ty$.

And $U(y_n) = \rho_\alpha(y_n, Ty_n)$ for all $\alpha \in (0, 1]$

$\lim_{y_n\to y} U(y_n) = \lim_{y_n\to y} \rho_\alpha(y_n, Ty_n)$ for all $\alpha \in (0, 1]$

$= \rho_\alpha\{\lim_{y_n\to y} (y_n, Ty_n)\}$

$= \rho_\alpha(y, Ty)$ for all $\alpha \in (0, 1]$

$= U(y)$

Also,

$U(T^{n-1}z) - U(T^n z) = \rho_\alpha(T^{n-1}z, T\,T^{n-1}z) - \rho_\alpha(T^n z, T\,T^n z)$ for all $\alpha \in (0, 1]$

Moreover,

$\rho_\alpha(T^n z, T^{n+1}z) \quad \prec \max\{\rho_\alpha(T^{n-1}z, T^n z), \rho_\alpha(T^n z, T^{n+1}z), \rho_\alpha(T^{n-1}z, T^n z)\}$

$\qquad < \rho_\alpha(T^{n-1}z, T^n z)$ for all $\alpha \in (0, 1]$

i.e. $U(T^n z) < U(T^{n-1}z)$ for all non-negative values of n.

Proceeding in this way we have

$U(T^n z) < U(T^{n-1}z) < U(T^{n-2}z) < \ldots < U(T^0 z = z)$ for $n \geq 0$.

Therefore $\{U(T^n z)\}$ is a convergent sequence and let $\lim_{n\to\infty} U(T^n z) = r$.

Also let $\{n_i\}$ be a sequence of positive integers such that $\lim_{n_i\to\infty} T^{n_i} z = \xi$.

Then we have $U(\xi) = U(\lim_{n_i\to\infty} T^{n_i} z)$

$\qquad = U(\lim_{n_i\to\infty} T^{n_i+1} z)$

$$= U(\lim_{n_i\to\infty} TT^{n_i}z)$$

$$= U(T\lim_{n_i\to\infty} T^{n_i}z)$$

$$= U(T\xi)$$

Again, $U(T\xi) = U(T\lim_{n_i\to\infty} T^{n_i}z)$

$$= U(T\lim_{n_i\to\infty} T^{n_i+1}z)$$

$$= U(TT\lim_{n_i\to\infty} T^{n_i}z)$$

$$= U(T^2\lim_{n_i\to\infty} T^{n_i}z)$$

$$= U(T^2T\lim_{n_i\to\infty} T^{n_i}z)$$

....

....

$= U(\lim_{n\to\infty} T^n z)$ for $n > n_i$

$$= \lim_{n\to\infty} U(T^n z)$$

$$= r$$

So, $U(\xi) = U(T\xi) = r$.

Now, for all $\alpha \in (0, 1]$, we have

$$\rho_\alpha(T\xi, \xi) = \rho_\alpha(T\xi, \lim_{n_i\to\infty} T^{n_i}z)$$

$$= \rho_\alpha(T\xi, \lim_{n_i\to\infty} T^{n_i+1}z)$$

$$= \rho_\alpha(T\xi, \lim_{n_i\to\infty} T^{n_i}z)$$

But, $\rho_\alpha(T\xi, T(T^{n_i}z)) < \max\{\rho_\alpha(\xi, T\xi), \rho_\alpha(T^{n_i}z, T^{n_i+1}z), \rho_\alpha(\xi, T^{n_i}z)\}$ for all $\alpha \in (0, 1]$

or, $\lim_{n_i \to \infty} \rho_\alpha(T\xi, T(T^{n_i}z)) \leq \lim_{n_i \to \infty} [\max\{\rho_\alpha(\xi, T\xi), \rho_\alpha(T^{n_i}z, T^{n_i+1}z), \rho_\alpha(\xi, T^{n_i}z)\}]$

or, $\rho_\alpha(T\xi, \lim_{n_i \to \infty} T^{n_i+1}z) \leq \max\{\rho_\alpha(\xi, T\xi), \rho_\alpha(\lim_{n_i \to \infty} T^{n_i}z, \lim_{n_i \to \infty} T^{n_i+1}z), \rho_\alpha(\xi, \lim_{n_i \to \infty} T^{n_i}z)\}$

Therefore, $\rho_\alpha(T\xi, \xi) \leq \max\{\rho_\alpha(\xi, T\xi), 0, \rho_\alpha(\xi, \xi)\}$

i.e. $\rho_\alpha(T\xi, \xi) \leq \max\{\rho_\alpha(\xi, T\xi), 0, 0\}$

i.e. $\rho_\alpha(T\xi, \xi) = 0$ for $\alpha \in (0, 1]$

i.e. $T\xi = \xi$.

Therefore, $r = \lim_{n \to \infty} U(T^n z)$

$= \lim_{n \to \infty} \rho_\alpha(T^n z, T^{n+1}z)$ for all $\alpha \in (0, 1]$

$= 0$ since $\xi = T\xi$

Now to prove that the sequence $\{T^n z\}$ converges to ξ. Given $\in > 0$, there exists a positive integer k such that

$\max\{U(T^k z), U(T^k z, \xi)\}\} < \in$ for all positive integer $n \geq k$.

Now, for all $\alpha \in (0, 1]$

$$\begin{aligned}
\rho_\alpha(T^n z, \xi) &= \rho_\alpha(T^n z, T^n \xi) \\
&= \rho_\alpha\{T(T^{n-1}z, T(T^{n-1}\xi)\} \\
&< \max\{\rho_\alpha(T^{n-1}z, T^n z), \rho_\alpha(T^{n-1}\xi, T^n \xi), \rho_\alpha(T^{n-1}\xi, T^{n-1}\xi) \\
&= \max\{U(T^{n-1}z), 0, \rho_\alpha(T^{n-1}z, \xi)\}
\end{aligned}$$

i.e. $\rho_\alpha(T^n z, \xi) < \max\{U(T^{n-1}z,), \rho_\alpha(T^{n-1}z, \xi)\}$

i.e. $\rho_\alpha(T^n z, \xi) < \max\{U(T^{n-2}z,), \rho_\alpha(T^{n-2}z, \xi)\}$ for all $\alpha \in (0, 1]$

so for $k \leq n$

$$\begin{aligned}
\rho_\alpha(T^n z, \xi) &< \max\{U(T^k z,), \rho_\alpha(T^k z, \xi)\} \text{ for all } \alpha \in (0, 1] \\
&< \in
\end{aligned}$$

Therefore, $\lim_{n \to \infty} \rho_\alpha(T^n z, \xi) = 0$ for all $\alpha \in (0, 1]$ and this completes the proof of the theorem.

REFERENCES

1. R. Badard, Fuzzy Sets and Systems 13 (1984), 291-302.
2. R. K. Bose and D. Sahani, Fuzzy Sets and Systems 21 (1987), 53-58.
3. A. Citra and P. V. Subrahmanyam, Jour. of Math. Anal. Appl. 124 (1987), 584-590.
4. P. Das and T. Basu, Bull. Cat. Math. Soc. 87 (1995), 313-318.
5. C. Felbin, Fuzzy Sets and Systems 48 (1992), 239-248.
6. S. Heilpern, Jour. of Math. Anal. Appl. 83(1981), 566-569.
7. M. C. Joshi and R. K. Bose, Some topics in non linear functional analysis; WILEY EASTERN LTD; p. 81-90.
8. J. S. Jung; Y. J. Cho and J. K. Kim: Fuzzy Sets and Systems 61 (1994) 199-207.
9. O. Kaleva, and S. Seikkala : Fuzzy Sets and Systems 12 (1984), 215-219.
10. T. Som, and R. N. Mukherjee : Fuzzy Sets and Systems 33 (1989), 213-219.
11. V. Sharma : Cal. Math. Soc. 86 (1994) 119-126.

Sikha Saikia (Bordoloye)
Department of Mathematics,
R. G. Baruah College,
GUWAHATI - 781025, Assam (India)

Jour. Pure Math., Vol. 21, 2004, pp. 65-74

PSEUDO ALGEBRAIC HOMEOMORPHISM

N. R. DAS AND G. C. DEKA

ABSTRACT : Notions of Pseudo algebraic spaces are introduced and studied in [1,4]. This paper is a continuation of the study concerning Pseudo algebraic homeomorphism and its p-Kernel with some of their basic properties. Finally Pseudo inverse of a set with the help of a Pseudo algebraic function on Pseudo algebraic space is introduced.

Key words : Pseudo topological space, Pseudo algebraic space, Pseudo algebraic homeomorphism, Pseudo open map, Pseudo continuous map, Pseudo Kernel and Pseudo inverse.

1. INTRODUCTION

Introducing notion of Pseudo algebraic structure, some of its basic properties are developed in [1] and [4]. In this note, notion of Pseudo homeomorphism and its p-Kernel are introduced. Notions of Pseudo universes are developed on p-a space. Topological behaviour and normal character of the p-Kernel are established.

2. PRELIMINARIES

We have reproduced some notion of Pseudo algebraic structure from [1] & [4] for easy understanding of the materials incorporated in subsequent sections.

Definition 2.1: Let X be a non-empty set and T a class of subsets of X such that

i) $X \in T$;

ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A$ for every $A \in T$;

and iii) the intersection of a finite collection of members of T is a member of T.

The class T is called a Pseudo topology and the pair (X, T) is called a p-topological space. Where there is no scope for confusion, X may be simply called p-topological space. The members of T are called p-open sets in X. A set A_0 is called a minimal p-open set. Uniqueness of A_0 is already established in [1] and [4].

Example 2.1 : A topological space is a p-topological space. The null set ϕ is the minimal p-open set.

Example 2.2 : A group is a p-topological space. The class of all subgroups of the group may be taken as a p-topology with the identity subgroup as the minimal p-open set.

3. PSEUDO ALGEBRAIC SPACES

Now we define a p-a space which is based on action of mapping $\alpha : P^x \times P^x \to P^x$ (P^x is the power set of X) upon the subsets of X.

Definition 3.1 : A p-topological space (X, T) is said to have a Pseudo algebraic structure (p-a structure) if there exists a Pseudo algebraic function (p-a function),

$\alpha : P^x \times P^x \to P^x$ (P^x is the power set of X)

satisfying the following conditions :

i) $\alpha(\alpha(A, B), C) = \alpha(A, \alpha(B, C))$, $(A, B, C \in P^x)$

ii) $\alpha(A, B) \in T$ if $\alpha(A, B) = \alpha(B, A)$, $(A, B \in T)$

iii) if $A_1 \subseteq A$, $B_1 \subseteq B$, then $\alpha(A_1, B_1) \subseteq \alpha(A, B)$

iv) $\alpha(A_0, A) = \alpha(A, A_0)$, ($A \in P^x$ and A_0 is the minimal p-open set).

The triplet (X, T, α) is called a Pseudo algebraic structure (p-a structure).

Example 3.1 : A topological space (X, T) is p-a space where $\alpha(A, B) = A \cup B$.

Example 3.2 : A group G with the usual p-topology T is a p-a space where $\alpha(A, B) = AB$ (AB means usual product of complexes A and B).

Definition 3.2 : A subset A in a p-a space (X, T, α) is called a p-normal subset if $\alpha(A, Y) = \alpha(Y, A)$, $\forall\, Y \in P^x$.

Definition 3.3 : The p-topology T of a p-a space (X, T, α) is said to be p-normal if every p-open set is p-normal and in this case, the p-a space (X, T, α) is said to be p-normal space.

4. PSEUDO CONTINUITY

Definition 4.1 : Let (X, T) and (Y, T^*) be two p-topological spaces and f be a mapping from X to Y. We say that the mapping f is p-continuous if $f^{-1}(A^*) \in T$ whenever $A^* \in T^*$ and is called p-open if $f(A) \in T^*$ whenever $A \in T$.

Proposition 4.1 : Let (X, T) and (Y, T^*) be two p-topological spaces with the minimal p-open sets A_0 and A_0^* respectively.

Then (i) f is p-continuous $\Rightarrow f(A_0) \subseteq A_0^*$

(ii) f is p-open and p-continuous $\Rightarrow f(A_0) = A_0^*$.

Proof : (i) If f is p-continuous, then $f^{-1}(A_0^*)$ is p-open subset of X and so $A_0 \subseteq f^{-1}(A_0^*)$ and hence, $f(A_0) \subseteq ff^{-1}(A_0^*) \subseteq A_0^*$.

(ii) If in addition to continuity f is also p-open, then $f(A_0)$ is a p-open subset of Y and so $A_0^* \subseteq f(A_0)$. This together with (i) proves that $f(A_0) = A_0^*$.

5. PSEUDO ALGEBRAIC HOMEOMORPHISM

Definition 5.1. A map $f : (X, T, \alpha) \to (Y, T^*, \beta)$ from a p-a space (X, T, α) into p-a space (Y, T^*, β) is said to be a p-a homeomorphism if

i) f is both p-open and p-continuous map

ii) $f(\alpha(A, B)) = \beta(f(A), f(B))$ $(A, B \in P^X)$

and iii) $\alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*, B^*))$ $(A^*, B^* \in P^Y)$

6. SOME TOPOLOGICAL PROPERTIES OF PSEUDO ALGEBRAIC HOMEOMORPHISM

Concerning p-a homeomorphism we have,

Proposition 6.1. A p-a homeomorphism maps the minimal p-open set onto the minimal p-open set.

Proof : Let $f : (X, T, \alpha) \to (Y, T^*, \beta)$ be a p-a homeomorphism from p-a space X onto p-a space Y. Let A_0 and A_0^* be the minimal elements of X and Y respectively.

$A_0 \subseteq f^{-1}(A_0^*)$, so $f(A_0) \subseteq ff^{-1}(A_0^*) \subseteq A_0^*$. Also $A_0^* \subseteq f(A_0)$

Hence $f(A_0) = A_0^*$.

Proposition 6.2. A p-a homeomorphism maps a p-normal p-open set onto a p-normal p-open set.

Proof : Let $f : (X, T, \alpha) \to (Y, T^*, \beta)$ be a p-a homeomorphism from one p-a space (X, T, α) to another p-a space (Y, T^*, β), Let A be a p-normal p-open subset of X. Then $f(A)$ is a p-normal p-open subset of Y.

Let N be any p-open subset of Y. Then from the p-continuity of f, it follows that $f^{-1}(N)$ is a p-open subset of X. Since A is a p-normal p-open subset of X,

$\alpha\ (A, f^{-1}\ (N)) = \alpha\ (f^{-1}\ (N), A)$ .... (i)

Since f is a p-a homeomorphism onto,

$f f^{-1}\ (N) = N$ and

$f\ (\alpha\ (A, f^{-1}\ (N))) = \beta\ (f\ (A), f f^{-1}\ (N)) = \beta\ (f\ (A), N)$

also $f\ (\alpha\ (f^{-1}\ (N), A)) = \beta\ (f f^{-1}\ (N), f\ (A)) = \beta\ (N. f\ (A))$

from (i), it follows that

$\beta\ (f\ (A), N) = \beta\ (N, f\ (A))$

Since N is arbitrary,

$\beta\ (f\ (A), N) = \beta\ (N, f\ (A))$ for every $N \in T^*$.

Thus $f\ (A)$ is a p-normal p-open subset of Y.

Example: 6.1: Let $T_1 = \{\{a\}, \{a, c\}, X\}$ be a p-topology on $X = \{a, b, c\}$ and $p^x = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Let $T_2 = \{\{p\}, Y\}$ be a p-topology on $Y = \{p, q\}$ and $p^y = \{\phi, \{p\}, \{q\}, Y\}$. The mapping defined as follows is a p-a homeomorphism.

$$f : X \to Y$$

$$\begin{tikzcd}[row sep=small] a \arrow[r] & p \\ b \arrow[r] & q \\ c \arrow[uur] & \end{tikzcd}$$

Remark 6.1. If $f : (X, T, \alpha) \to (Y, T^*, \beta)$ is a p-a homeomorphism and Y_0 a p-normal p-open subset of Y, then it can be verified with the help of the following example that $f^{-1}\ (Y_0)$ is not necessarily a p-nolmal p-open subset of X.

Example 6.2. Let $T_1 = \{\{a\}, \{a, c\}, X)$ be a p-topology on $X = \{a, b, c\}$ and $T_2 = \{\{p\}, Y\}$ be a p-topology on $Y = \{p, q\}$. The mapping defined as follows is a p-a homeomorphism.

$$f : X \to Y,\ f_0 : T_1 \to T_2.$$

$$\begin{tikzcd}[row sep=small] a \arrow[r] & p \\ b \arrow[r] & q \\ c \arrow[uur] & \end{tikzcd}$$

Where f_0 is an induced mapping induced by f such that $f_0\ (A_0) = A_0^*$, $f_0\ (X) = Y$; A_0, A_0^* are the minimal p-open subsets of X and Y respectively.

Let $\alpha : P^X \times P^X \to P^X$ (P^X is the power set of X), Then α is a p-a function which satisfies the following conditions :

i) $\alpha\ (\alpha\ (\{a\}, \{b\}), \{c\}) = \alpha\ (\{a\} \cup \{b\}, \{c\}), (\{a\}, \{b\}, \{c\} \in P^X)$

$$= (\{a\} \cup \{b\}) \cup \{c\}$$

$$= \{a\} \cup (\{b\} \cup \{c\})$$

$$= \{a\} \cup \alpha\ (\{b\}, \{c\})$$

$$= \alpha\ (\{a\}, \alpha\ (\{b\}, \{c\})$$

Similarly we can show for other members of P^X.

ii) $\alpha\ (\{a\}, \{a, c\}) = \{a\} \cup \{a, c\} = \{a, c\} \cup \{a\} = \alpha\ (\{a, c\}, \{a\}), (\{a\}, \{a, c\} \in T)$

$\therefore\ \alpha\ (\{a\}, \{a, c\}) \in T.$

ii) $\{a\} \subset \{a, b\}$, $\{b\} \subset \{b, c\}$, then

$\alpha\ (\{a\}, \{b\}) = \{a\} \cup \{b\} \subseteq \{a, b\} \cup \{b, c\}$

$= \alpha\ (\{a, b\}, \{b, c\}), (\{a\}, \{b\}, \{a, b\}, \{b, c\} \in P^X)$

iv) $\alpha\ (\{a\}, \{b, c\}) = \{a\} \cup \{b, c\} = \{b, c\} \cup \{a\} = \alpha\ (\{b, c\}, \{a\})$

1. $f\ (\{a\}) = \{p\}, f\ (\{a, c\}) = \{p\}, f\ (X) = Y$

$\therefore$ f is p-open

$f^{-1}\ (\{p\}) = \{a, c\} \in T_1, f^{-1}\ (Y) = X$

$\therefore$ f is p-continuous.

f is p-open and p-continuous.

2. $A = \{a\}$, $B = \{b\}$

$f\ (\alpha\ (A, B)) = f\ (\alpha\ (\{a\}, \{b\})) = f\ (\{a\} \cup \{b\}) = f\ \{a, b\} = \{p, q\}$

$\beta\ (f\ (A), f\ (B)) = \beta\ (\{p\}, \{q\}) = \{p\} \cup \{q\} = \{p, q\}$

$\therefore$ $f\ (\alpha\ (A, B)) = \beta\ (f\ (A), f\ (B))\ (A, B \in P^X)$

$\alpha\ (f^{-1}\ (A^*), f^{-1}\ (B^*)) = \alpha\ (\{a, c\}, \{b\}) = \{a, c\} \cup \{b\} = X, (A^* = \{p\}, B^* = \{q\})$

$$f^{-1}(\beta(A^*, B^*)) = f^{-1}(\beta(\{p\}, \{q\}) = f^{-1}(\{p\} \cup \{q\})$$
$$= f^{-1}(\{p, q\}) = \{a, b, c\} = X$$
$$\alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*, B^*)) \quad (A^*, B^* \in P^Y)$$

$\therefore$ f is a p-a homeomorphism.

Again let $\beta : P^Y \times P^Y \to P^Y$ (P^Y is the power set of Y). Then β is also p-a function which can be proved easily.

Clearly f_0 is not one-one since

$f_0(\{a\}) = f_0(\{a, c\}) = \{p\} \in T_2$

In Table-2, $\{p\}$ is p-normal p-open set in T_2. But $f^{-1}(\{p\}) = \{a, c\}$ which is not p-normal p-open set in T_1, since $\alpha(\{a, c\}, b\}) \neq \alpha(\{b\}, \{a, c\})$, [From Table-1]

Table-1

α	ϕ	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	X
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	X
$\{a\}$	ϕ	$\{a\}$	$\{a, b\}$	$\{a, c\}$	$\{a, b\}$	$\{a, c\}$	X	X
$\{b\}$	ϕ	$[a, b\}$	$\{b\}$	$\{a\}$	$\{b\}$	$\{b\}$	$\{c\}$	X
$\{c\}$	ϕ	$\{a, c\}$	$\{b, c\}$	$\{c\}$	$\{c\}$	$\{a\}$	$\{b\}$	X
$\{a, b\}$	ϕ	$\{a, b\}$	$\{a\}$	$\{c\}$	$\{a, b\}$	X	X	X
$\{a, c\}$	ϕ	$\{a, c\}$	$\{a\}$	$\{b\}$	X	$\{a, c\}$	X	X
$\{b, c\}$	ϕ	X	$\{a\}$	$\{c\}$	$\{a, b\}$	$\{c\}$	$\{b, c\}$	X
X	X	X	X	X	X	X	X	X

Table -2

β	ϕ	$\{p\}$	$\{q\}$	Y
ϕ	ϕ	ϕ	$\{p\}$	Y
$\{p\}$	ϕ	$\{p\}$	Y	Y
$\{q\}$	ϕ	Y	$\{q\}$	Y
Y	Y	Y	Y	Y

The result given in remark 6.1 holds if f is one-one. We prove this in the following proposition.

Proposition 6.3 : If the *p-a* homeomorphism f induces a one-one correspondence between the *p*-open subsets of X and the *p*-open subsets of Y, then for any *p*-normal *p*-open subset Y_0 of Y, $f^{-1}(Y_0)$ is *p*-normal in X.

Proof. Under the given condition, for any $A \in T$ there exists an A^* in T^* such that $A = f^{-1}(A^*)$. Then $\beta(Y_0, A^*) = \beta(A^*, Y_0)$ for all $A^* \in T^*$, $f^{-1}(Y_0)$ is a *p*-open subset of X and for any A $\in$ T,

We have

$$\begin{aligned}\alpha(f^{-1}(Y_0), A) &= \alpha(f^{-1}(Y_0), f^{-1}(A^*)\\ &= f^{-1}(\beta(Y_0, A^*)\\ &= f^{-1}(\beta(A^*, Y_0) \text{ (since } Y_0 \text{ is } p\text{-normal)}\\ &= \alpha(f^{-1}(A^*), f^{-1}(Y_0))\\ &= \alpha(A, f^{-1}(Y_0))\end{aligned}$$

Therefore, $f^{-1}(Y_0)$ is *p*-normal in (X, T, α).

7. PSEUDO KERNEL OF P-A HOMEOMORPHISM

Definition 7.1. Let $f : (X, T, \alpha) \to (Y, T^*, \beta)$ be a *p-a* homeomorphism from one *p-a* space (X, T, α) to another *p-a* space (Y, T^*, β). Let A_0^* be the minimal *p*-open subset of Y. Then $(f^{-1}(A_0^*)$ is called the *p*-Kernel of f.

Example 7.1. Let $T_1 = \{\{p\}, \{p, q\}, X\}$ be a p-topology on $X_1 = \{p, q, r\}$ where $P^{X1} = \{\phi, \{p\}, \{q\}, \{r\}, \{p, q\}, \{p, r\}, \{q, r\}, X_1\}$ and let $T_2 = \{\{a\}, X_2\}$ be a p-topology on $X_2 = \{a, b\}$ where $P^{X2} = \{\phi, \{a\}, \{b\}, X_2\}$. The mapping defined as follows is a p-a homeomorphism.

$$f : X_1 \to X_2$$

$$\begin{tikzcd}
p \arrow[r] & a \\
c \arrow[ur] & \\
r \arrow[r] & b
\end{tikzcd}$$

Again $f^{-1}(A_0^*) = f^{-1}(\{a\}) = \{p, q\}$ is the p-Kernel of f where $A_0^* = \{a\} \in T_2$.

We conclude this section with the following

Proposition 7.1. If $f : (X, T, \alpha) \to (Y, T^*, \beta)$ is a p-a homeomorphism from one p-a space (X, T, α) to another p-a space (Y, T^*, β) and A is p-Kernel, then A is p-normal in X if f is one-one.

Proof. From proposition 6.2, we know that

$\beta(A_0^*, N) = \beta(N, A_0^*)$, for every $N \subseteq Y$, where

A_0^* is the minimal p-open subset of Y and $A = f^{-1}(A_0^*)$

for $M \subseteq X$,

$f^{-1}(\beta(A_0^*, N) = f^{-1}(\beta(N, A_0^*)), N = f(M)$

$\Rightarrow \alpha(f^{-1}(A_0^*), f^{-1}(N)) = \alpha(f^{-1}(N), f^{-1}(A_0^*))$

$\Rightarrow \alpha(A, f^{-1}(f(M))) = \alpha(f^{-1}(f(M)), A)$

$\Rightarrow \alpha(A, M) = \alpha(M, A)$

$\therefore A$ is p-normal in X.

8. PSEUDO INVERSE

Now we discuss Pseudo inverse of a set on p-a space as follows :

Definition 8.1. Let (X, T, α) be a p-a space with Pseudo algebraic function

$$\alpha : P^X \times P^X \sim P^X.$$

$B \in P^X$ is said to be p-inverse of A where $A \in P^X$ if $\alpha(A, B) = A_0 = \alpha(B, A)$ where A_0 is the minimal p-open set. Then we write $B = A^{-1}$.

Proposition 8.1. Let (X, T, α) and (Y, T^*, β) be two p-a spaces and $f : X \to Y$ be an onto p-a homeomorphism.

Then $f(A^{-1}) = (f(A))^{-1}$

Proof. We observe that

$$\beta(f(A^{-1}), f(A)) = f(\alpha(A^{-1}, A)) = f(A_0) = A_0^*$$

and $$\beta(f(A), f(A^{-1})) = f(\alpha(A, A^{-1})) = f(A_0) = A_0^*$$

This shows that

$$f(A^{-1}) = (f(A))^{-1}.$$

The following is the generalization on the algebraic property $(ab)^{-1} = b^{-1}a^{-1}$.

Proposition 8.2. Let (X, T, α) be a p-a space with $\alpha : P^X \times P^X \to P^X$ (P^X is the power set of X) satisfying

$$\alpha(A_0, A_0) = A_0.$$

Let $A, B \in P^X$ and A^{-1} and B^{-1} be the p-inverses of A and B respectively.

Then $(\alpha(A, B))^{-1} = \alpha(B^{-1}, A^{-1})$

Proof. $\alpha(\alpha(A, B), \alpha(B^{-1}, A^{-1}))$

$$= \alpha(\alpha(\alpha(A, B), B^{-1}), A^{-1})$$
$$= \alpha(\alpha(A, \alpha(B, B^{-1})), A^{-1})$$
$$= \alpha(\alpha(A, A_0), A^{-1})$$
$$= \alpha(\alpha(A_0, A), A^{-1})$$
$$= \alpha(A_0, \alpha(A, A^{-1}))$$
$$= \alpha(A_0, A_0)$$
$$= A_0$$

This proves that

$$(\alpha(A, B))^{-1} = \alpha(B^{-1}, A^{-1}).$$

REFERENCES

1. Das, N. R. and Deka, G. C. : *Sub Pseudo Algebraic Spaces,* Journal Pure Math. Vol. 20, 2003. PP. 17-23.
2. Deka, G. C. : *Product Pseudo Algebraic Spaces,* Journal Pure Math. Vol. 19, 2002, PP. 73-79.
3. Kelley, J. L. : *General Topology,* First Edition, Van Nostrand Reinhold Company, 1955.
4. Mena, B. N. and Deka, G. C. : *Pseudo Algebraic Spaces II,* Published by Allahabad Mathematical Society, Second Biennial Conference, (April 7-9, 1990), Symposium on Topology.
5. Munkres, James. : *Topology, a first course,* Prentice Hall, 1975.
6. Willard, S. : *General Topology,* Addition Wesley Publishing Company inc.. mass, 1986.

Department of Mathematics,
Gauhati University;
Guwahati-781014, Assam; India.

Department of Mathematics,
Kamrup College,
Chamata-781306,
Nalbari; Assam; India.

Jour. Pure Math., Vol. 21, 2004, pp. 75-80

QUOTIENT PSEUDO ALGEBRAIC SPACES

N. R. DAS AND G. C. DEKA

ABSTRACT : Pseudo algebraic structure is a mixed structure arising out of a Pseudo topological structure and an algebraic structure. Quotient Pseudo algebraic space is introduced with the help of a p-normal subspace in a natural way. Finally we obtain a natural p-a homomorphism on a quotient p-a space.

Key Words : Pseudo topological spaces, Quotient Pseudo topological spaces, Quotient Pseudo algebraic spaces, Pseudo algebraic homomorphism are denoten by p-topological spaces, Quotient p-topological spaces, Quotient p-a spaces and p-a homomorphism.

1. INTRODUCTION

In this paper, we have studied Pseudo algebraic spaces, a mixed Pseudo algebraic topological structure arising out of Pseudo topological structure and algebraic structure. Notion of Pseudo continuity and p-a homomorphism are introduced in a natural way. Quotient Pseudo algebraic space is developed with the help of a Pseudo algebraic p-normal subspace. Finally a Pseudo algebraic homomorphism is obtained between a p-a space and a Quotient Pseudo algebraic space.

2. PRELIMINARIES

A Pseudo topological space can be regarded as a near topological space. In this structure, existence of null set and closeness under arbitrary union properties of a topological space are developed. Weakness of this structure is recovered by introducing the existence of a minimal element and a new notion of Pseudo topological space is developed. Mixing this Pseudo topological structure with algebraic structure the concept of a Pseudo algebraic space is developed.

Definition 2.1 : A family T of subsets of a non empty set X is said to be a Pseudo topology (p-topology) on X if

(i) $X \in T$.

(ii) there exists an $A_0 \in T$ such that $A_0 \subseteq A \;\; \forall \;\; A \in T$;

and (iii) the intersection of a finite collection of members of T is a member of T.

The pair (X, T) is called a Pseudo topological space (p-topological space). The members of T are called p-open sets.

A topological space is a trivial example of p-topological space. The null set Φ is the minimal p-open set in this case.

One can see easily that the minimal element in a p-topological space is unique.

A group $(G, .)$ is another example of p-topological space. In this case, the collection of all subgroups of G is a p-topology and the identity subgroup is the minimal p-open set. Similarly subrings of a ring, subfields of a field and subspaces of a vector space form Pseudo topologies on ring, field and vector space respectively.

Now we define a Pseudo algebraic space which is based on actions of a mapping $\alpha : p^X \times p^X \to p^X$ upon the subsets of X.

Definition 2.2 : A p-topological space (X, T) is said to, have a Pseudo algebraic structure (p-a structure) if there exists a function,

$$\alpha : p^X \times p^X \to p^X$$

satisfying the following conditions :

(i) $\alpha(\alpha(A, B), C) = \alpha(A, \alpha(B, C))$ for all $A, B, C, \in p^X$.

(ii) $\alpha(A, B) \in T$ if and only if $\alpha(A, B) = \alpha(B, A)$ for $A, B \in T$.

(iii) $A_1 \subseteq A$ and $B_1 \subseteq B$ iff $\alpha(A_1, B_1) \subseteq \alpha(A, B)$ for all $A_1, B_1, A, B \in P^X$.

(iv) $\alpha(A_0, A) = \alpha(A, A_0)$ for every $A \in p^X$, where A_0 is the minimal p-open set.

The triplet (X, T, α) is said to be a Pseudo algebraic topological structure or simply a Pseudo algebraic space (p-a space).

Definition 2.3 : A subset A of a p-a space (X, T, α) is said to be a p-normal set if $\alpha(A, B) = \alpha(B, A)$, $\forall\ B \in p^X$.

Definition 2.4 : The p-topology T of a p-a space (X, T, α) is said to be p-normal if every p-open set is p-normal. If the p-topology of a p-a space is p-normal then p-a space is said to be a p-normal p-a space.

Definition 2.5 : Let Y be a non empty subset of a p-topological space X. The p-topology T' on Y defined by

$T' = \{A \cap Y : A \in T\}$ is called the relative p-topology on Y and the p-topological space (Y, T') is called a sub p-topological space of (X, T).

Definition 2.6 : A sub p-topological space (Y, T') of a p-topological space (X, T) is called a sub p-a space of p-a space (X, T, α) with a p-a structure α' if α induces a p-a function α' on p^Y such that

(i) $\alpha'(A, B) = \alpha(A, B)$ for $A, B, \in p^Y$,

(ii) $\alpha'(A, B) \in T'$ if and only if $\alpha'(A, B) = \alpha'(B, A)$ for $A, B \in T'$

and (iii) $\alpha'(A_0', A) = \alpha'(A, A_0')$ for $A \in p^Y$ and the minimal p-open subset A_0' of (Y, T').

Definition 2.7 : A mapping f from a p-topological space (X, T) into a p-topological space (Y, T^*) is said to be

(i) p-continuous if $f^{-1}(A^*) \in T$ for every $A^* \in T^*$.

(ii) p-open if $f(A) \in T^*$ for every $A \in T$.

Definition 2.8 : A mapping f from a p-a space (X, T, α) to (Y, T^*, β) is said to be a p-a homomorphism if the following are satisfied.

(i) f is both p-open and p-continuous.

(ii) $f(\alpha(A, B)) = \beta(f(A), f(B))$ for $A, B \in p^X$.

and (iii) $\alpha(f^{-1}(A^*), f^{-1}(B^*)) = f^{-1}(\beta(A^*, B^*))$ for $A^*, B^* \in P^Y$.

3. QUOTIENT PSEUDO ALGEBRAIC SPACES

In this section, we introduce the notion of quotient p-topology and quotient p-a space. Subsequently a p-a homomorphism is obtained in this structures.

Let Y be a p-normal subset of a p-normal p-a space (X,T, α).

Put $\frac{X}{Y} = \{\alpha(\{X\}, Y) : x \in X\}$ and

$T_Y = \{\alpha(\{x\}, Y) : x \in A \in T\} = \{\alpha(A, Y) : A \in T\}$.

Proposition 3.1 : $(\frac{X}{Y}, T_Y)$ is a Pseudo topological space.

Proof : Clearly, $\frac{X}{Y} = \{\alpha(\{x\}, Y) : x \in X \in T\}$ is in T_Y.

Let A_0 be the minimal p-open subset of (X, T).

Then for every $A \in T$, $\alpha(A_0, Y) = \{\alpha(\{x\}, Y) : x \in A_0\} \subseteq \{\alpha(\{x\}, Y) : x \in A\} = \alpha(A, Y)$.

This shows that $\alpha(A_0, Y)$ is the minimal element of T.

For a finite collection $\{(A_i, Y) : 1 \leq i \leq n\}$ of subsets of $\frac{X}{Y}$,

$$\bigcap_{i=1}^{n} (A_i, Y) = \{\alpha(\{x\}, Y) : x \in A \in T, 1 \leq i \leq n\}$$

$$=\{\alpha(\{x\}, Y) : x \in \bigcap_{i=1}^{n} A_i \in T\}$$

Hence, $\bigcap_{i=1}^{n} (A_i, Y) \in T_Y$ and $(\frac{X}{Y}, T_Y)$ is a p-topological space.

The p-topological space $(\frac{X}{Y}, T_Y)$ is called the quotient p-topological space.

Proposition 3.2 : Let Y be a p-normal subset of p-a space $(X,T,\ \alpha)$.We define

$\alpha^{*} : P^{X/Y} \times P^{X/Y} \longrightarrow P^{X/Y}$ by

$\alpha^{*}(\alpha(A,\ Y),\ \alpha(B,\ Y)) = \alpha(\alpha(A,\ B),\ Y)$ where $A,\ B \in T$.

$= \alpha\{\alpha\ (\{x\},\ Y) : x \in \alpha(A,\ B)\}$

Let $T_Y = \{\alpha(\{x\},\ Y) : x \in A \in T\} = \{\alpha(A,\ Y) : A \in T\}$

Then $(\frac{X}{Y},\ T_Y,\ \alpha^{*})$ is a p-a space.

Proof : As seen in Proposition 3.1, T_Y is a p-topology on $\frac{X}{Y}$. Let us see that α^{*} is well defined.

Suppose

$\alpha(A_1,\ Y) = \alpha(A_2,\ Y),\ A_1,\ A_2 \in T$

$\alpha(B_1,\ Y) = \alpha(B_2,\ Y),\ B_1,\ B_2 \in T.$

Then $\alpha^{*}(\alpha(A_1,\ Y),\ \alpha(B_1,\ Y)) = \alpha(\alpha(A_1,\ B_1),\ Y) = \alpha(A_1,\ \alpha(B_1,\ Y))$

$= \alpha(A_1,\ \alpha(B_2,\ Y))$

$= \alpha(A_1,\ \alpha(Y,\ B_2))$ (since Y is p-normal)

$= \alpha(\alpha(A_1,\ Y),\ B_2)$

$= \alpha(\alpha(A_2,\ Y),\ B_2)$

$= \alpha(A_2,\ \alpha(Y,\ B^2))$

$= \alpha(A_2,\ \alpha(B_2,\ Y))$

$= \alpha(\alpha(A_2,\ B_2),\ Y)$

$= \alpha^{*}\ (\alpha(A_2,\ Y),\ \alpha(B_2,\ Y))$

Hence α^{*} is well defined.

Although verifications that $(\frac{X}{Y},\ T_Y,\ \alpha^{*})$ is a p-a space is routine, for sake of completion, we explain as follows,

$$\begin{aligned}
&\alpha^{*}(\alpha^{*}(\alpha(A,\ Y),\ \alpha(B,\ Y)),\ \alpha(C,\ Y)),\ A,\ B,\ C \in T\\
&= \alpha^{*}(\alpha(\alpha(A,\ B),\ Y),\ \alpha(C,\ Y))\\
&= \alpha(\alpha((\alpha(A,\ B),\ C),\ Y))\\
&= \alpha(\alpha(A,\ \alpha(B,\ C)),\ Y)\\
&= \alpha^{*}(\alpha(A,\ Y),\ \alpha(\alpha(B,\ C),\ Y))\\
&= \alpha^{*}(\alpha(A,\ Y),\ \alpha^{*}(\alpha(B,\ Y),\ \alpha(C,\ Y)))
\end{aligned}$$

Next, if $\alpha^*(\alpha(A, Y), \alpha(B, Y)) = \alpha^*(\alpha(B, Y), \alpha(A, Y))$

Then $\alpha(\alpha(A, B), Y) = \alpha(\alpha(B, A), Y)$ $A, B \in T$

Using p-normal character of Y.

$$\alpha(Y, \alpha(A, B)) = \alpha(\alpha(A, B), Y)$$

$$\therefore \alpha^*(\alpha(A, Y), \alpha(B, Y) \in T_Y$$

Let $\alpha(A_1, Y) \subseteq \alpha(A, Y)$

$\alpha(B_1, Y) \subseteq \alpha(B, Y)$ Then $A_1 \subseteq A$ & $B_1 \subseteq B$ and

$$\alpha(A_1, B_1) \subseteq \alpha(A, B)$$

$$\begin{aligned}\alpha^*(\alpha(A_1, Y), \alpha(B_1, Y)) &= \alpha(\alpha(A_1, B_1), Y)\\ &\subseteq \alpha(\alpha(A, B), Y)\\ &= \alpha^*(\alpha(A, Y), \alpha(B, Y))\end{aligned}$$

Finally,

$$\begin{aligned}&\alpha^*(\alpha(A_0, Y), \alpha(A, Y)), A_0, A \in T\\ &= \alpha(\alpha(A_0, A), Y)\\ &= \alpha(\alpha(A, A_0), Y), \quad A \in T \text{ and } A \text{ is } p\text{-normal.}\\ &= \alpha^*(\alpha(A, Y), \alpha(A_0, Y))\end{aligned}$$

The space $(\frac{X}{Y}, T_Y, \alpha^*)$ is called a p-a space. It is called a quotient p-a space.

The following result shows the existence of a natural p-a homomorphism.

Proposition 3.3 : Let Y be a p-normal subset of the p-normal p-a space (X,T, α) and $(\frac{X}{Y}, T_Y, \alpha^*)$ be the quotient p-a space. Then the natural mapping

$f : (X, T) \to (\frac{X}{Y}, T_Y)$ defined by

$f(x) = \alpha(\{x\}, Y)$ is a p-a homomorphism.

Proof : Let $\alpha(A, Y)$ be a p-open subset of $(\frac{X}{Y}, T_Y)$. We recall that

$T_Y = \{\alpha(A, Y) : A \in T\} = \{\alpha(\{x\}, Y) : x \in A \in T\}$. Then

$f^{-1}(\alpha(A, Y)) = \{x \in A \in T\} = A \in T$. Hence f is p-continuous.

Again for $A \in T$,

$f(A) = \alpha(A, Y) \in T_Y$, Hence f is p-open.

$$f(\alpha(A, B)) = \alpha(\alpha(A, B), Y)\ A, B \in P^X$$

$$= \alpha^*(\alpha(A, Y), \alpha(B, Y))$$

$$= a^*(f(A), f(B))$$

and $\alpha(f^{-1}(\alpha(A, Y), f^{-1}(\alpha(B, Y)) = \alpha(A, B) = f^{-1}(\alpha(\alpha(A, B), Y))$

$$= f^{-1}(\alpha^*(\alpha(A, Y), \alpha(B, Y)))$$

for $\alpha(A, Y), \alpha(B, Y) \in P^Y$.

$\therefore$ f is a p-a homomorphism.

We have discussed a few structural properties of a p-a space (X, T, α). In place of X considering group, ring, field and vector space structure, such structural properties can be enriched.

REFERENCES

1. Das, N. R and Deka, G.C : Sub Pseudo Algebraic Spaces, to appear in Journal Pure Math. Vol 20, 2004.
2. Deka, G. C. : Product Pseudo Algebraic Spaces, Journal Pure Math. Vol. 19, 2002, PP 73-79.
3. Kelley, J. L. : General Topology, First Edition, Van Nostrand Reinhold Company, 1955.
4. Mena, B. N. and Deka, G. C. : Pseudo Algebraic Spaces II, Published by Allahabad Mathematical Society, Second Biennial Conference (April 7-9, 1990), Symposium on Topology.
5. Munkres, James : Topology, a first course, Prentice Hall, 1975.
6. Willard, S : General Topology, Addition Wesley Publishing Company inc. mass, 1968.

Department of Mathematics,
Gauhati University
Guwahati - 781 014,
Assam; India.

Department of Mathematics,
Kamrup College
Chamata - 781 306, Nalbari,
Assam; India.

Jour. Pure Math., Vol. 21, 2004, pp. 81-85

A PSEUDO $\tilde{W}_2$ CURVATURE TENSOR ON A RIEMANNIAN MANIFOLD

(Dedicated to Prof. M. C. Chaki on his 90th birth anniversary)

B. PRASAD AND ASHWAMEDH MOURYA

ABSTRACT : Recently Prasad (2002) introduced pseudo projective curvature tensor in a Riemannian manifold. In this paper we define a pseudo $\tilde{W}_2$ curvature tensor on a Riemannian manifold and obtain its several properties.

1. INTRODUCTION

The pseudo $\tilde{W}_2$ curvature tensor $\tilde{W}_2$ on a Riemannian manifold (M^n, g) $(n > 2)$ of type (1, 3) is defined as follows.

$$\tilde{W}_2 (X, Y, Z) = aR(X, Y, Z) + b[g(Y, Z)QX - g(X, Z)QY] \tag{1.1}$$

$$-\frac{r}{n}[\frac{a}{n-1} + b] [g(Y, Z)X - g(X, Z)Y]$$

where a and b are constants such that $a, b \neq 0$, R is the curvature tensor, S is the Ricci tensor, r is the scalar curvature and Q is the (1, 1) Ricci tensor defined by

$$g(QX, Y) = S(X, Y) \text{ for all } X \text{ and } Y.$$

This paper deals with Riemannian manifolds (M^n, g) $(n > 2)$ for which $\tilde{W}_2$ is conservative [1].

A manifold (M^n, g) $(n > 2)$ shall be called pseudo $\tilde{W}_2$ flat or Pseudo $\tilde{W}_2$ conservative according as $\tilde{W}_2 = 0$ or div $\tilde{W}_2 = 0$. In this paper it is proved that pseudo $\tilde{W}_2$ flat manifold is of constant curvature provided that $(a - b) \neq 0$. Further it is proved that every (M^n, g) of constant curvature is pseudo $\tilde{W}_2$ flat. Finally a necessary and sufficient condition for an (M^n, g) to be pseudo $\tilde{W}_2$ conservative is obtained.

It can be easily verified that

(1.2) $\tilde{W}_2(X, Y, Z) = -\tilde{W}_2(Y, X, Z)$

(1.3) $\tilde{W}_2(X, Y, Z) + \tilde{W}_2\{Y, Z, X) + \tilde{W}_2(Z, X, Y) = 0$

If $a = 1$ and $b = -\frac{1}{n-1}$, then (1. 1) takes the form.

$$\tilde{W}_2(X, Y, Z) = R(X, Y, Z) - \frac{1}{n-1} [g(Y, Z)\, QX - g(X, Z)\, QY] = W_2(X, Y, Z)$$

where W_2 is the curvature tensor [2].

Hence the curvature tensor W_2 is a particular case of the tensor $\tilde{W}_2$. For this reason $\tilde{W}_2$ is called pseudo $\tilde{W}_2$ curvature tensor.

2. PSEUDO $\tilde{W}_2$ FLAT MANIFOLD

In this section we assume that $\tilde{W}_2 = 0$. Then from (1.1) we get,

(2.1) $aR(X, Y, Z) + b[g(Y, Z)QX - g(X, Z)QY] - \frac{r}{n} [\frac{a}{n-1} + b] [g(Y, Z)X - g(X, Z)Y] = 0$

From (2.1) we get,

(2.2) $a'R(X, Y, Z, W) + b[g(Y, Z)S(X, W) - g(X, Z)S(Y, W]$

$$-\frac{r}{n}[\frac{a}{n-1} + b] [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] = 0$$

where

(2.3) $'R(X, Y, Z, W) = g.(R(X, Y, Z);W)$

Putting $X = W = e_i$ in (2. 2) where $\{e_i\}$ is an orthonormal basis of the tangent space at a point of the manifold and taking summation over e_i, $1 \leq i \leq n$, we get,

(2.4) $(a - b)\, [S(Y, Z) - \frac{r}{n}\, g(Y, Z)] = 0$

If $a - b \neq 0$ then from (2.4) we get,

(2,5) $S(Y, Z) = \frac{r}{n}\, g(Y, Z)$

Hence, in view of (2.2) and (2.5), we get,

(2.6) $'R(X, Y, Z, W) = \frac{r}{n(n-1)} [g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$, for $a \neq 0$

Thus, we can state the following theorem

Theorem 1. *A pseudo* $\tilde{W}_2$ *flat manifold is a manifold of constant curvature, provided that* $a - b \neq 0$ From (1) we have

$$(2.7) \quad '\tilde{W}_2(X, Y, Z, W) = a'R(X, Y, Z, W) + b[g(Y, Z)S(X, W) - g(X, Z)S(Y, W)]$$
$$-\frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)g(X, W) - g(X, Z)\, g(Y, W)]$$

where $'\tilde{W}_2\,(X, Y, Z, W) = g(\tilde{W}_2(X, Y, Z), W)$

Now (2.7) can be written as follows

$$(2.8) \quad '\tilde{W}_2(X, Y, Z, W) = a['R(X, Y, Z, W) - \frac{r}{n(n-1)}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}]$$
$$+ b[g(Y, Z)\{S\{X, W) - \frac{r}{n}g(X, W)\} - g(X, Z)\{S(Y, W) - \frac{r}{n}g(Y, W)\}]$$

If an (M^n, g) $(n > 2)$ is a manifold of constant curvature then it is an Einstein manifold. In view of this the equation (2.8) gives

$$'\tilde{W}_2(X, Y, Z, W) = 0$$

Hence we can state the following theorem.

Theorem 2. *A manifold* (M^n, g) $(n > 2)$ *of constant curvature is pseudo* $\tilde{W}_2$ *flat.*

3. Pseudo $\tilde{W}_2$ Conservative (M^n, g) $(n > 2)$

In this section we assume that

(3.1) $\operatorname{div} \tilde{W}_2 = 0$

where div denotes divergence.

Now differentiating (1 .1) covariantly, we get,

$$(D_U \tilde{W}_2)(X, Y, Z) = a(D_U R)(X, Y, Z) + b[g(Y, Z)(D_U Q)X - g(X, Z)(D_U Q)Y]$$

$$-\frac{(\mathrm{Dur})}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)X - g(X, Z)Y]$$

which gives contraction.

$$(3.2) \quad (\mathrm{div}\ \tilde{W}_2)(X, Y, Z) = a(\mathrm{div}\ R)(X, Y, Z) + b[g(Y, Z)(\mathrm{div}\ Q)X - g(X, Z)(\mathrm{div}\ Q)Y]$$

$$-\frac{1}{n}\left[\frac{a}{n-1} + b\right][g(Y, Z)\, dr\,(X) - g(X, Z)\, dr\,(Y)]$$

But from [3], we have

$$(\mathrm{div}\ R)(X, Y, Z) = (D_X S)(Y, Z) - (D_Y S)(X, Z)$$

Hence (3.2) gives

$$(3.3) \quad (\mathrm{div}\ \tilde{W}_2)(X, Y, Z) = a[(D_X S)(Y, Z) - (D_Y S)(X, Z)] + \left[\frac{b(n-1)(n-2) - 2a}{2n(n-1)}\right]$$

$$[g(Y, Z) dr(X) - g(X, Z) dr\,(Y)]$$

It the Ricci tensor $S(X, Y)$ be of Codazzi type, i.e.

$$(D_X S)(Y, Z) = (D_Y S)(X, Z)$$

then from, (3.3) we get

$$(3.4) \quad (\mathrm{div}\ \tilde{W}_2)(X, Y, Z) = \left[\frac{b(n-1)(n-2) - 2a}{2n(n-1)}\right][g(Y, Z) dr\,(X) - g(X, Z)\, dr\,(Y)]$$

Hence if div $\tilde{W}_2 = 0$, then

$$g(Y, Z)\, dr\,(X) - g(X, Z)\, dr\,(Y) = 0$$

which shows that r is constant. Again if r is constant then from (3.4) we get,

$$(\mathrm{div}\ \tilde{W}_2\ (X, Y, Z) = 0$$

Hence we can state the following theorem

Theorem 3. *If in a Riemannian manifold $(M^n. g)$ $(n > 2)$ the Ricci tensor is codazzi type then the manifold is pseudo $\tilde{W}_2$ conservative if and only if the scalar curvature is constant.*

Further we have form (3.3),

(3.5) $$\frac{(\mathrm{div}\tilde{W}_2)(X,Y,Z)}{a} = [(D_X S)(Y, Z) - (D_Y S)(X, Z)]\left[\frac{b(n-1)(n-2)-2a}{2an(n-1)}\right][g(Y, Z)dr(X) - g(X, Z)\, dr(Y)]$$

Since we assume that $a \neq 0$

Then from (3.5) we can state the theorem as follows

Theorem 4. *If in a Riemannian manifold (M_n, g) $(n > 2)$ the pseudo $\tilde{W}_2$ curvature tensor is such that $a \neq 0$ then the manifold is pseudo $\tilde{W}_2$ curvature if and only if*

$$(D_X S)(Y, Z) - (D_Y S)(X, Z) = \left[\frac{b(n-1)(n-2)-2a}{2an(n-1)}\right][g(Y, Z)dr(X) - g(X, Z)\, dr(Y)]$$

REFERENCES

1. N. J. Hicks : Notes on a differential geometry Affiliated East West Press Pvt. Ltd.
2. G. P. Pokhariyal and R. S. Mishra : Curvature tensor and their relativistic significance Yokohama math jour vol. 18, no. 2 1970, 105-108.
3. L. P. Eisenhat : Riemannian geormetry Princtan Uni. Press 1926, 82-91.
4. B. Prasad : A pseudo projective curvature tensor on a Riemannian manifold Bull. Cal. Math. Soc., vol 94(3), 2002, 163-166.

Department of Mathematics,
S. M. M. Town (P. G.) College,
Ballia (U. P.) - 277 001,
India.
e-mail : bp_kushwaha@sify.com

Jour. Pure Math., Vol. 21, 2004, pp. 87-100

FIXED POINT THEOREMS FOR MULTI-VALUED MAPS IN D-METRIC SPACES

M. S. RATHORE[1], DEEPAK SINGH[2], NAVAL SINGH[3]

ABSTRACT : In this paper we establish a fixed point theorem for three multi-valued maps in D-metric spaces.

1. INTRODUCTION

Nadler [4] first initiated to prove fixed point theorems of multi-valued contraction mappings in complete metric spaces. After the works of Nadler [4] many research workers have devoted their much time to prove fixed point theorems of multi-valued mappings in various ways e.g. Bose and Mukhetjee[l], Garegnani and Massa [3] etc, used concept od Hausdorff metric H(A, B). Further some fixed point theorems for multi-valued contraction mappings have been obtained by Veerapandi and Rao [5]. They also proved a result on fixed point of orbitally continuous and multivalued mappings of complete and bounded D-metric spaces.

2. PRELIMINARIES

We recall some definitions introduced by Dhage [2].

Definition 2.1. Let X be a non empty set. A generalized metric (or D-metric) on X is a function D from $X \times X \times X$ to R^+ (= the set of non negative real numbers) that satisfies the following conditions

(i) $D(x, y, z) = 0$ if and only if $x = y = z$. (Sufficiency),

(ii) $D(x, y, z) = D(y, x, z) = \ldots$ (Symmetry),

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all x, y, z, a in X (Rectangle inequality).

The pair (X, D) is then called a generalized metric space (or D-metric space).

Geometrically the D-metric $D(x, y, z)$ represents the perimeter of a triangle whose vertices are x, y and z.

Definition 2.2. A sequence $\{x_n\}$ of points in a D-metric space (X, D) is said to be D-convergent

to the point $x \in X$ if, for each $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$D(x_n, x_m, x) < \varepsilon, \qquad \text{for all } m, n \geq n_0.$$

Definition 2.3. A sequence $\{x_n\}$ of points in a D-metric space (X, D) is said to be D-Cauchy if for each $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$D(x_n, x_m, x_p) < \varepsilon, \qquad \text{for all } m, p > n \geq n_0.$$

Definition 2.4. A D-metric space (X, D) is said to be complete if every D-Cauchy sequence in X converges to a point in X.

Definition 2.5. The D-metric space X is said to be D-bounded if there exists $M > 0$, such that $D(x, y, z) \leq M$ for all x, y, z in X

We state some more definitions and result.

Definition 2.6. If $B(X)$ is the collection of all non-empty bounded subsets of a D-metric space (X, D), and for $A, B, C \in B(X)$

Let $H(A, B, C) = \sup \{D(a, b, c) : a \in A, b \in B, c \in C\}$. Then

(i) $H(A, B, C) \geq 0$ for $A, B, C \in B(X)$ and $H(A, B, C) = 0$ implies that $A = B = C$ with a singleton; further if $A = B = C$, then $H(A, B, C)$ is the perimeter of the largest triangle contained in the set $A > 0$, otherwise A is singleton,

(ii) $H(A, B, C) = H(B, C, A) = H(C, A, B)$ for $A, B, C \in B(X)$,

(iii) $H(A, B, C) \leq H(A, B, E) + H(A, E, C) + H(E, B, C)$

for $A, B, C, E \in B(X)$.

Definition 2.7. Let (X, D) be a D-metric space and $CB(X)$ be the set of all bounded closed subsets of X. Let $T : X \to CB(X)$. T is said to be a multi-valued contraction mapping if $H(Tx, Ty, Tz) \leq q\, D(x, y, z)$ for all $x, y, z \in X$, where $0 \leq q < 1$, is a fixed real number.

Definition 2.8. A map $T : X \to CB(X)$ is said to be orbitally continuous at x if $x_n \to x$ as $n \to \infty$ in X with respect to D implies $\{x_n\}$ has a subsequence $\{x_{ni}\}$ such that $Tx_{ni} \to Tx$ with respect to H as $i \to \infty$. If T is orbitally continuous at all points of X, then it is orbitally continuous on X.

Veerapandi and Rao [5] proved the following.

Theorem 2.9. Let (X, D) be a complete bounded D-metric space.

If $T : X \to CB(X)$ is multi valued contraction mapping, then T has a fixed point.

3. MAIN RESULT

We generalize Theorem 2.9 of [5] in the following theorem.

Theorem 3.1. Let (X, D) be a complete and bounded D-metric space, let $S, T, P : X \to CB(X)$ be a multi-valued and orbitally continuous mapping such that S, T and P are pair-wise disjoint self-mappings of X, satisfying

(i) $$H(Sx, Ty, Pz) \le a_1\, D^*(x, y, z) + a_2\, \{D^*(Sx, y, z) + D^*(x, Sx, z)\} + a_3\, \{D^*(y, z, pz) + D^*(Sx, Ty, z)\} + a_4\, \{D^*(Sx, y, Pz) + D^*(x, Ty, Pz)\} + \frac{a_5\, \{D^*(x, Sx, z)\, D^*(y, Ty, z) + D^*(y, Sx, z)\, D^*(x, Ty, z)\}}{D(x, y, z)}$$

for all distinct x, y, z in X and non-negative reals a_1, a_2, a_3, a_4 and as such that

$$(a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5) < 1 \text{ with } (1/2) < (\text{arp})^{J+1} < 1$$

where $a = \dfrac{(a_1 + 2a_2 + a_4 + a_5)}{(1 - 2a_3 - 2a_4 - a_5)}$, $\quad r = \dfrac{(a_1 + 2a_2 + 2a_3 + a_4 + a_5)}{(1 - a_3 - a_4 - a_5)}$,

$$p = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_3 - 2a_4 - a_5)}$$

where $D^*(A, B, C) = \inf \{D(a, b, c);\ a \in A,\ b \in B,\ c \in C\}$ for $A, B, C \in CB(X)$. Then S, T and P have a unique common fixed point in X.

Proof : Let $x_0 \in X$ be arbitrary. Consider a sequence $\{x_n\}$ in X such that

$$X_{3n+1} \in Sx_{3n}, \; x_{3n+2} \in E\; Tx_{3n+1} \text{ and } x_{3n+3} \in Px_{3n+2}, \text{ for all } n = 0, 1, 2, 3 \ldots$$

Now using (i), we have

$$\begin{aligned} D(x_1, x_2, x_3) &\le H(Sx_0, Tx_1, Px_2) \\ &\le a_1\, D^*(x_0, x_1, x_2) + a_2\, \{D^*(Sx_0, x_1, x_2) + D^*(x_0, Sx_0, x_2)\} \\ &\quad + a_3\, \{D^*(x_1, x_2, Px_2) + D^*(Sx_0, Tx_1, Px_2)\} \\ &\quad + a_4\, \{D^*(Sx_0, x_1, Px_2) + D^*(x_0, Tx_1, Px_2)\} \\ &\quad + \frac{a_5\, \{D^*(x_0, Sx_0, x_2)\, D^*(x_1, Tx_1, x_2) + D^*(x_1, Sx_0, x_2)\, D^*(x_0, Tx_1, x_2)\}}{D(x_0, x_1, x_2)} \\ &\le a_1\, D(x_0, x_1, x_2) + a_2\, \{D(x_1, x_1, x_2) + D(x_0, x_1, x_2)\} \end{aligned}$$

$$+ a_3 \{D(x_1, x_2, x_3) + D(x_1, x_2, x_2)\}$$
$$+ a_4 \{D(x_1, x_1, x_3) + D(x_1, x_2, x_3)\}$$
$$+ \frac{a_5 \{D(x_0, x_1, x_2) D(x_1, x_2, x_2) + D(x_1, x_1, x_2) D(x_0, x_2, x_2)\}}{D(x_0, x_1, x_2)}$$

$$\leq a_1 D(x_0, x_1, x_2) + a_2 \{D(x_0, x_1, x_2) + D(x_0, x_1, x_2)\}$$
$$+ a_3 \{D(x_1, x_2, x_3) + D(x_1, x_2, x_3)\}$$
$$+ a_4 \{D(x_1, x_2, x_3) + D(x_1, x_2, x_3) + D(x_0, x_1, x_3) + D(x_0, x_2, x_1)\}$$
$$+ a_5 \{D(x_1, x_2, x_3) D(x_0, x_1, x_2)$$

This Implies

$$D(x_1, x_2, x_3) \leq \frac{(a_1 + 2a_2 + a_4 + a_5)}{(1 - 2a_3 - 2a_4 - a_5)} D(x_0, x_1, x_2) + \frac{a_4}{(1 - 2a_3 - 2a_4 - a_5)} D(x_0, x_1, x_3)$$

or $D(x_1, x_2, x_3) \leq a\, D(x_0, x_1, x_2) + b\, D(x_0, x_1, x_3)$ (1)

where $a = \dfrac{(a_1 + 2a_2 + a_4 + a_5)}{(1 - 2a_3 - 2a_4 - a_5)}$, $b = \dfrac{a_4}{(1 - 2a_3 - 2a_4 - a_5)}$.

Now using (i), we have

$$D(x_1, x_2, x_3) \leq H(Tx_1, Px_2, Sx_3) = H(Sx_3, Tx_1, Px_2)$$
$$\leq a_1 D^*(x_3, x_1, x_2) + a_2 \{D^*(Sx_3, x_1, x_2) + D^*(x_3, Sx_3, x_2)\}$$
$$+ a_3 \{D^*(x_1, x_2, Px_2) + D^*(Sx_3, Tx_1, x_2)\}$$
$$+ a_4 \{D^*(Sx_3, x_1, Px_2) + D^*(x_3, Tx_1, Px_2)\}$$
$$+ \frac{a_5 \{D^*(x_3, Sx_3, x_2) D^*(x_1, Tx_1, x_2) + D^*(x_1, Sx_3, x_2) D^*(x_3, Tx_1, x_2)\}}{D(x_3, x_1, x_2)}$$

$$\leq a_1 D(x_1, x_2, x_3) + a_2 \{D(x_4, x_1, x_2) + D(x_3, x_4, x_2)\}$$
$$+ a_3 \{D(x_1, x_2, x_3) + D(x_4, x_2, x_2)\}$$
$$+ a_4 \{D(x_4, x_1, x_3) + D(x_3, x_2, x_3)\}$$
$$+ \frac{a_5 \{D(x_3, x_4, x_2) D(x_1, x_2, x_2) + D(x_1, x_4, x_2) D(x_3, x_2, x_2)\}}{D(x_3, x_1, x_2)}$$

$$\leq a_1 D(x_1, x_2, x_3) + a_2 \{D(x_1, x_2, x_4) + D(x_2, x_3, x_4)\}$$

$$+ a_3 \{D(x_1, x_2, x_3) + D(x_2, x_3, x_4)\}$$
$$+ a_4 \{D(x_2, x_3, x_4) + D(x_1, x_2, x_4) + D(x_1, x_3, x_2) + D(x_2, x_3, x_4)\}$$
$$+ a_5 \{D(x_2, x_3, x_4) D(x_1, x_2, x_4)$$

This implies that

$$D(x_2, x_3, x_4) \leq \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_3 - 2a_4 - a_5)} D(x_1, x_2, x_3) + \frac{(a_2 + a_4 + a_5)}{(1 - a_2 - a_3 - 2a_4 - a_5)} D(x_1, x_2, x_4)$$

or $D(x_2, x_3, x_4) \leq p\ D(x_1, x_2, x_3) + q\ D(x_1, x_2, x_4)$

where $p = \dfrac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_3 - 2a_4 - a_5)}$, $q = \dfrac{(a_2 + a_4 + a_5)}{(1 - a_2 - a_3 - 2a_4 - a_5)}$.

using (1), above becomes

$$D(x_2, x_3, x_4) \leq ap\ D(x_0, x_1, x_2) + bp\ D(x_0, x_1, x_3) + q\ D(x_1, x_2, x_4) \tag{2}$$

Now using (i), we have

$$D(x_3, x_4, x_5) \leq H(Px_2, Sx_3, Tx_4) = H(Sx_3, Tx_4, Px_2)$$
$$\leq a_1 D^*(x_3, x_4, x_2) + a_2 \{D^*(Sx_3, x_4, x_2) + D^*(x_3, Sx_3, x_2)\}$$
$$+ a_3 \{D^*(x_4, x_2, Px_2) + D^*(Sx_3, Tx_4, x_2)\}$$
$$+ a_4 \{D^*(Sx_3, x_4, Px_2) + D^*(x_3, Tx_4, Px_2)\}$$
$$+ \frac{a_5 \{D^*(x_3, Sx_3, x_2) D^*(x_4, Tx_4, x_2) + D^*(x_4, Sx_3, x_2) D^*(x_3, Tx_4, x_2)\}}{D(x_3, x_4, x_2)}$$
$$\leq a_1 D(x_2, x_3, x_4) + a_2 \{D(x_4, x_4, x_2) + D(x_3, x_4, x_2)\}$$
$$+ a_3 \{D(x_4, x_2, x_3) + D(x_4, x_5, x_2)\}$$
$$+ a_4 \{D(x_4, x_4, x_3) + D(x_3, x_5, x_3)\}$$
$$+ \frac{a_5 \{D(x_3, x_4, x_2) D(x_4, x_5, x_2) + D(x_4, x_4, x_2) D(x_3, x_5, x_2)\}}{D(x_3, x_4, x_2)}$$
$$\leq a_1 D(x_2, x_3, x_4) + a_2 \{D(x_2, x_3, x_4) + D(x_2, x_3, x_4)\}$$
$$+ a_3 \{D(x_2, x_3, x_4) + D(x_3, x_4, x_5)\}$$
$$+ a_4 \{D(x_2, x_3, x_4) + D(x_3, x_4, x_5)\}$$
$$+ a_5 \{D(x_3, x_4, x_5) D(x_2, x_3, x_5) + D(x_2, x_4, x_3) + D(x_3, x_5, x_2)\}$$

This implies that

$$D(x_3, x_4, x_5) \leq \frac{(a_1+2a_2+2a_3+a_4+a_5)}{(1-a_3-a_4-a_5)} D(x_2, x_3, x_4) + \frac{(a_3+2a_5)}{(1-a_3-a_4-a_5)} D(x_2, x_3, x_5)$$

or $D(x_3, x_4, x_5) \leq r\ D(x_2, x_3, x_4) + s\ D(x_2, x_3, x_5)$

where $r = \dfrac{(a_1+2a_2+2a_3+a_4+a_5)}{(1-a_3-a_4-a_5)}$, $\qquad s = \dfrac{(a_3+2a_5)}{(1-a_3-a_4-a_5)}$.

using (2), above becomes

$$\begin{aligned} D(x_3, x_4, x_5) \leq{} & \text{apr}\ D(x_0, x_1, x_2) + \text{bpr}\ D(x_0, x_1, x_3) + \text{qr}\ D(x_1, x_2, x_4) \\ & + \text{s}\ D(x_2, x_3, x_5) \end{aligned} \tag{3}$$

similarly,

$$\begin{aligned} D(x_4, x_5, x_6) \leq{} & a^2pr\ D(x_0, x_1, x_2) + abpr\ D(x_0, x_1, x_3) + aqr\ D(x_1, x_2, x_4) \\ & + as\ D(x_2, x_3, x_5) + b\ D(x_3, x_4, x_6) \end{aligned} \tag{4}$$

$$\begin{aligned} D(x_5, x_6, x_7) \leq{} & a^2p^2r\ D(x_0, x_1, x_2) + abp^2r\ D(x_0, x_1, x_3) + apqr\ D(x_1, x_2, x_4) \\ & + aps\ D(x_2, x_3, x_5) + bp\ D(x_3, x_4, x_6) + q\ D(x_4, x_5, x_7) \end{aligned} \tag{5}$$

$$\begin{aligned} D(x_6, x_7, x_8) \leq{} & a^2p^2r^2\ D(x_0, x_1, x_2) + abp^2r^2\ D(x_0, x_1, x_3) + apqr^2\ D(x_1, x_2, x_4) \\ & + aprs\ D(x_2, x_3, x_5) + bpr\ D(x_3, x_4, x_6) + qr\ D(x_4, x_5, x_7) \\ & + s\ D(x_5, x_6, x_8) \end{aligned} \tag{6}$$

$$\begin{aligned} D(x_7, x_8, x_9) \leq{} & a^3p^2r^2\ D(x_0, x_1, x_2) + a^2bp^2r^2\ D(x_0, x_1, x_3) + a^2pqr^2\ D(x_1, x_2, x_4) \\ & + a^2prs\ D(x_2, x_3, x_5) + abpr\ D(x_3, x_4, x_6) + aqr\ D(x_4, x_5, x_7) \\ & + as\ D(x_5, x_6, x_8) + b\ D(x_6, x_7, x_9) \end{aligned}$$

$$\begin{aligned} D(x_8, x_9, x_{10}) \leq{} & a^3p^3r^2\ D(x_0, x_1, x_2) + a^2bp^3r^2\ D(x_0, x_1, x_3) + a^2p^2qr^2\ D(x_1, x_2, x_4) \\ & + a^2p^2rs\ D(x_2, x_3, x_5) + abp^2r\ D(x_3, x_4, x_6) + apqr\ D(x_4, x_5, x_7) \\ & + aps\ D(x_5, x_6, x_8) + bp\ D(x_6, x_7, x_9) + q\ D(x_7, x_8, x_{10}) \end{aligned}$$

$$\begin{aligned} D(x_9, x_{10}, x_{11}) \leq{} & a^3p^3r^3\ D(x_0, x_1, x_2) + a^2bp^3r^3\ D(x_0, x_1, x_3) \\ & + a^2p^2qr^3\ D(x_1, x_2, x_4) + a^2p^2r^2s\ D(x_2, x_3, x_5) \\ & + abp^2r^2\ D(x_3, x_4, x_6) + apqr^2\ D(x_4, x_5, x_7) + aprs\ D(x_5, x_6, x_8) \\ & + bpr\ D(x_6, x_7, x_9) + qr\ D(x_7, x_8, x_{10}) + s\ D(x_8, x_9, x_{11}) \end{aligned}$$

Continuing in this way, we have

$$
\begin{aligned}
D(x_{3n-2}, x_{3n-1}, x_{3n}) \le\ & (apr)^{n-1} \{a\ D(x_0, x_1, x_2) + b\ D(x_0, x_1, x_3)\} \\
& + (apr)^{n-2} \{aqr\ D(x_1, x_2, x_4) + as\ D(x_2, x_3, x_5) + b\ D(x_3, x_4, x_6)\} \\
& + (apr)^{n-3}\{aqr\ D(x_4, x_5, x_7) + as\ D(x_5, x_6, x_8) + b\ D(x_6, x_7, x_9)\} \\
& \ldots \qquad\qquad \ldots \\
& \ldots \qquad\qquad \ldots \\
& + (apr)^0 \{aqr\ D(x_{3n-5}, x_{3n-4}, x_{3n-2}) + as\ D(x_{3n-4}, x_{3n-3}, x_{3n-1}) \\
& + b\ D(x_{3n-3}, x_{3n-2}, x_{3n})\} \\
& = (apr)^{n-1}\{a\ D(x_0, x_1, x_2) + \mathrm{b}\ D(x_0, x_1, x_3)\} \\
& \quad \sum_{k=0}^{n-2}(apr)^k \ \{aqr\ D(x_{3n-3k-5}, x_{3n-3k-4}, x_{3n-3k-2}) \\
& \quad + as\ D(x_{3n-3k-4}, x_{3n-3k-3}, x_{3n-3k-1}) + b\ D(x_{3n-3k-3}, x_{3n-3k-2}, x_{3n-3k})\} \\
& \quad \le (apr)^{n-1}\ (a + b)\ M + \sum_{k=0}^{n-2}(apr)^k\ (aqr + as + b)\ M,
\end{aligned}
$$

since D is bounded

$$= (apr)^{n-1}\ M\ \alpha_1 + \sum_{k=0}^{n-2}(apr)^k\ M\ \beta_1$$

where $\alpha_1 = (a + b)$, $\beta_1 = (aqr + as + b)$

Similarly

$$D(x_{3n-1}, x_{3n}, x_{3n+1}) \le (apr)^{n-1}\ M\ (ap + bp + q) + \sum_{k=0}^{n-2}(apr)^k\ M\ (aps + bp + q)$$

$$= (apr)^{n-1}\ M\ \alpha_2 + \sum_{k=0}^{n-2}(apr)^k\ M\ \beta_2$$

where $\alpha_2 = (ap + bp + q)$, $\beta_2 = (aps + bp + q)$ and

$$D(x_{3n}, x_{3n+1}, x_{3n+2}) \le (apr)^n\ M + \sum_{k=0}^{n-1}(apr)^k\ M\ (bpr + qr + s)$$

$$= (apr)^n\ M + \sum_{k=0}^{n-1}(apr)^k\ M\ \alpha_3$$

Where $\alpha_3 = (bpr + qr + s)$.

Now we shall prove that $\{x_n\}$ is a D-Cauchy sequence.

Taking m of the form $3k + 1$ $(k = 0, 1, 2, \ldots)$, we have for positive integer t and l.

$$D(x_m, x_{m+t}, x_{m+t+l}) \leq D(x_m, x_{m+1}, x_{m+t}) + D(x_m, x_{m+1}, x_{m+t+l}) + D(x_{m+1}, x_{m+t}, x_{m+t+l})$$

$$\leq (apr)^{(m-1)/3} M \alpha_1 + \sum_{k=0}^{(m-4)/3} (apr)^k M \beta_1 + (apr)^{(m-1)/3} M \alpha_1$$

$$+ \sum_{k=0}^{(m-4)/3} (apr)^k M \beta_1 + D(x_{m+1}, x_{m+2}, x_{m+t})$$

$$+ D(x_{m+1}, x_{m+2}, x_{m+t+l}) + D(x_{m+2}, x_{m+t}, x_{m+t+l})$$

$$\leq 2M\alpha_1 (apr)^{(m-1)/3} + 2M\beta_1 \sum_{k=0}^{(m-4)/3} (apr)^k + 2M\alpha_2 (apr)^{(m-1)/3}$$

$$+ 2M\beta_2 \sum_{k=0}^{(m-4)/3} (apr)^k + D(x_{m+2}, x_{m+3}, x_{m+t})$$

$$\leq 2M\alpha_1 (apr)^{(m-1)/3} + 2M\beta_1 \sum_{k=0}^{(m-4)/3} (apr)^k + 2M\alpha_2 (apr)^{(m-1)/3}$$

$$+ 2M\beta_2 \sum_{k=0}^{(m-4)/3} (apr)^k + 2M (apr)^{(m+2)/3}$$

$$+ 2M\alpha_3 \sum_{k=0}^{(m-1)/3} (apr)^k + D(x_{m+3}, x_{m+t}, x_{m+t+l})$$

$$\ldots \qquad \ldots$$

$$\ldots \qquad \ldots$$

$$\leq 2M\alpha_1 [(apr)^{(m-1)/3} + (apr)^{(m+2)/3} + \ldots + (apr)^{(m+t-5)/3}]$$

$$+ 2M\beta_1 [\sum_{k=0}^{(m-4)/3} (apr)^k + \sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+t-8)/3} (apr)^k]$$

$$+ 2M\alpha_2 [(apr)^{(m-1)/3} + (apr)^{(m+2)/3} + \ldots + (apr)^{(m+t-5)/3}]$$

$$+ 2M\beta_2 \left[\sum_{k=0}^{(m-4)/3} (apr)^k + \sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+t-8)/3} (apr)^k \right]$$

$$+ 2M \left[(apr)^{(m+2)/3} + (apr)^{(m+5)/3} + \ldots + (apr)^{(m+t-2)/3} \right]$$

$$+ 2M\alpha_3 \left[\sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+2)/3} (apr)^k + \sum_{k=0}^{(m+t-5)/3} (apr)^k \right]$$

$$+ D(x_{m+t-1}, x_{m+t}, x_{m+t+l})$$

$$\leq 2M\alpha_1 \left[(apr)^{(m-1)/3} + (apr)^{(m+2)/3} + \ldots + (apr)^{(m+t-5)/3} \right.$$

$$\left. + 1/2\ (apr)^{(m+t-2)/3} \right]$$

$$+ 2M\beta_1 \left[\sum_{k=0}^{(m-4)/3} (apr)^k + \sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+t-8)/3} (apr)^k \right.$$

$$\left. + 1/2 \sum_{k=0}^{(m+t-5)/3} (apr)^k \right]$$

$$+ 2M\alpha_2 \left[(apr)^{(m-1)/3} + (apr)^{(m+2)/3} + \ldots + (apr)^{(m+t-5)/3} \right]$$

$$+ 2M\beta_2 \left[\sum_{k=0}^{(m-4)/3} (apr)^k + \sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+t-8)/3} (apr)^k \right]$$

$$+ 2M \left[(apr)^{(m+2)/3} + (apr)^{(m+5)/3} + \ldots + (apr)^{(m+t-2)/3} \right]$$

$$+ 2M\alpha_3 \left[\sum_{k=0}^{(m-1)/3} (apr)^k + \sum_{k=0}^{(m+2)/3} (apr)^k + \sum_{k=0}^{(m+t-5)/3} (apr)^k \right]$$

since $(1/2) < (apr)^{J+1} < 1$, therefore it is easy to see

$$1 + (apr) + (apr)^2 + \ldots + (apr)^J < (apr)^{J+1} + (apr)^{J+2} + \ldots$$

i.e. $\sum_{k=0}^{J} (apr)^k < \sum_{k=J+1}^{\infty} (apr)^k$

Therefore

$$D(x_m, x_{m+t}, x_{m+t+l}) < 2M\alpha_1 \sum_{k=(m+t+1)/3}^{\infty} (apr)^k + 2M\beta_1 \left[\sum_{k=(m-1)/3}^{\infty} (apr)^k + \sum_{k=(m+2)/3}^{\infty} (apr)^k \right.$$

$$+ \ldots + \sum_{k=(m+t-5)/3}^{\infty} (apr)^k + \sum_{k=(m+t-2)/3}^{\infty} (apr)^k] + 2M\alpha_2 \sum_{k=(m+t-2)/3}^{\infty} (apr)^k$$

$$+ 2M\beta_2 [\sum_{k=(m-1)/3}^{\infty} (apr)^k + \sum_{k=(m+2)/3}^{\infty} (apr)^k + \ldots + \sum_{k=(m+t-5)/3}^{\infty} (apr)^k]$$

$$+ 2M \sum_{k=(m+t+1)/3}^{\infty} (apr)^k + 2M\alpha_3 [\sum_{k=(m+2)/3}^{\infty} (apr)^k + \sum_{k=(m+5)/3}^{\infty} (apr)^k$$

$$+ \ldots + \sum_{k=(m+t-2)/3}^{\infty} (apr)^k]$$

$\to 0$ as $m \to \infty$, since $apr < 1$, so each term in summation tends to zero as $m \to \infty$.

Thus $\{x_n\}$ is a D-Cauchy sequence. But X is complete so $\{x_n\}$ conveges to some $u \in X$.

i.e. $\lim\limits_{n \to \infty} x_n = u.$

We shall prove that u is a common fixed point of P, S and T. Since P, S, and T are orbitally continuous so the sequences.

$\{Sx_{3n}\}$, $\{Tx_{3n+1}\}$ and $\{Px_{3n+2}\}$ converge to Su, Tu and Pu respectively.

As $x_{3n+1} \in Sx_{3n}$, $x_{3n+2} \in Tx_{3n+1}$ and $x_{3n+3} \in Px_{3n+2}$ for all n, it follows that $u \in Su$, $u \in Tu$ and $u \in Pu$.

Hence u is a common fixed point of P, S and T.

To prove the uniqueness, let v be another fixed point of P, S and T.

Then

$$D(u, u, v) \le H(Su, Tu, Pv)$$

$$\le a_1 D^*(u, u, v) + a_2 \{D^*(Su, u, v) + D^*(u, Su, v)\}$$

$$+ a_3\{D^*(u, v, Pv) + D^*(Su, Tu, v)\} + a_4\{D^*(Su, u, Pv) + D^*(u, Tu, Pv)\}$$

$$+ \frac{a_5\{D^*(u, Su, v)\, D^*(u, Tu, v) + D^*(u, Su, v)\, D^*(u, Tu, v)\}}{D(u, u, v)}$$

$$\leq a_1 D(u, u, v) + a_2 \{D(u, u, v) + D(u, u, v)\} + a_3\{D(u, v, v) + D(u, u, v)\} + a_4\{D(u, u, v) + D(u, u, v)\} + \frac{a_5\{D(u, u, v) D(u, u, v) + D(u, u, v) D(u, u, v)\}}{D(u, u, v)}$$

or, $(1 - a_1 - 2a_2 - 2a_3 - 2a_4 - 2a_5) D(u, u, v) \leq 0$.

This implies that $u = v$. Therefore S, T and P have unique common fixed point in X.

Remark: If $S = T = P$ and $a_2 = a_3 = a_4 = a_5 = 0$. Then we get Theorem 2.9 of Veernpandi and Rao, K. C. [5].

Theorem 3.2. Let (X, D) be a complete and bounded D-metric space, let $\{S_n\}$, $\{T_n\}$, $\{P_n\}$, $:X \to CB(X)$ be a multi-valued and orbitally continuous mapping such that S_n, T_n and P_n are pair-wise disjoint self-mappings of X, satisfying

(i) $$H(S_{i'} x, T_{j'} y, P_{k'} z) \leq a_1 D^*(x, y, z) + a_2\{D^*(S_{i'} x, y, z) + D^*(x, S_{i'} x, z)\} + a_3\{D^*(y, z, P_{k'} z) + D^*(S_{i'} x, T_{j'} y, z)\} + a_4\{D^*(S_{i'} x, y, P_{k'} z) + D^*(x, T_{j'} y, P_{k'} z)\} + \frac{a_5\{D^*(x, S_{i'} x, z) D^*(y, T_{j'} y, z) + D^*(y, S_{i'} x, z) D^*(x, T_{j'} y, z)\}}{D(x, y, z)}$$

for all distinct x, y, z in X ; i', j', $k' \in N$ and non-negative reals a_1, a_2, a_3, a_4 and a_5 such that

$(a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5) < 1$ with $(1/2) < (arp)^{J+1} < 1$

where $a = \dfrac{(a_1 + 2a_2 + a_4 + a_5)}{(1 - 2a_3 - 2a_4 - a_5)}$, $\quad r = \dfrac{(a_1 + 2a_2 + 2a_3 + a_4 + a_5)}{(1 - a_3 - a_4 - a_5)}$

$$p = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_3 - 2a_4 - a_5)}$$

and where $D^*(A, B, C) = \inf \{D(a, b, c); a \in A, b \in B, c \in C\}$ for $A, B, C \in CB(X)$.

Then S_n, T_n and P_n have unique common fixed point in X.

Proof : Let $x_0 \in X$ be arbitrary. Consider a sequence $\{x_n\}$ in X such that

$x_{3n-2} \in S_{3k''+1}\, x_{3n-3}$, $x_{3n-1} \in T_{3k''+2}\, x_{3n-2}$ and $x_{3n} \in P_{3k''+3}\, x_{3n-1}$, for all $n = 1, 2, 3, ...$ and $k'' = 0, 1, 2, 3, ...$

Now using (i), we have

$$D(x_1, x_2, x_3) \leq H\,(S_1x_0, T_2x_1, P_3x_2)$$
$$\leq a_1\, D^*(x_0, x_1, x_2) + a_2\, \{D^*(S_1x_0, x_1, x_2) + D^*(x_0, S_1x_0, x_2)\}$$
$$+ a_3\, \{D^*(x_1, x_2, P_3x_2) + D^*(S_1x_0, T_2x_1, x_2)\}$$
$$+ a_4\, \{D^*(S_1x_0, x_1, P_3x_2) + D^*(x_0, T_2x_1, P_3x_2)\}$$
$$+ \frac{a_5\, \{D^*(x_0, S_1x_0, x_2)\, D^*(x_1, T_2x_1, x_2) + D^*(x_1, S_1x_0, x_2)\, D^*(x_0, T_2x_1, x_2)\}}{D(x_0, x_1, x_2)}$$
$$\leq a_1\, D(x_0, x_1, x_2) + a_2\, \{D(x_1, x_1, x_2) + D(x_0, x_1, x_2)\}$$
$$+ a_3\, \{D(x_1, x_2, x_3) + D(x_1, x_2, x_2)\}$$
$$+ a_4\, \{D(x_1, x_1, x_3) + D(x_0, x_2, x_3)\}$$
$$+ \frac{a_5\, \{D(x_0, x_1, x_2)\, D(x_1, x_2, x_2) + D(x_1, x_1, x_2)\, D(x_0, x_2, x_2)\}}{D(x_0, x_1, x_2)}$$

Now with the similar treatment as in Theorem 3.1, we can show $\{x_n\}$ is a D-cauchy sequence in X.

Next We shall prove that u is a common fixed point of P_n, S_n and T_n. Since P_n, S_n, and T_n are orbitally continuous so the sequences.

$\{S_nx_{3n}\}$, $\{T_nx_{3n+1}\}$ and $\{P_nx_{3n+2}\}$ converge to S_nu, T_nu and P_nu respectively, and so their subsequences $\{S_{3k''+1}\, x_{3n-3}\}$, $\{T_{3k''+2}\, x_{3n-2}\}$ and $\{P_{3k''+3}\, x_{3n-1}\}$ converge to S_nu, T_nu and P_nu respectively.

As $x_{3n-2} \in S_{3k''+1}\, x_{3n-3}$, $x_{3n-1} \in T_{3k''+2}\, x_{3n-2}$ and $x_{3n} \in P_{3k''+3}\, x_{3n-1}$, for all n, it follows that $u \in S_nu$, $u \in T_nu$ and $u \in P_nu$.

Hence u is a common fixed point of $\{S_j'\}$, $\{T_j'\}$ and $\{P_k'\}$.

To prove the uniqueness, let v be another fixed point.

Then

$$D\,(u, u, v) \leq H(S_i'u, T_j'u, P_k'v)$$
$$\leq a_1\, D^*(u, u, v) + a_2\, \{D^*(S_i'u, u, v) + D^*(u, S_i'u, v)\}$$
$$+ a_3\{D^*(u, v, P_k'v) + D^*(S_i'u, T_j'u, v)\} + a_4\{D^*(S_i'u, u, P_k'v) + D^*(u, T_j'u, P_k'v)\}$$
$$+ \frac{a_5\{D^*(u, S_i'u, v)\, D^*(u, T_j'u, v) + D^*(u, S_i'u, v)\, D^*(u, T_j'u, v)\}}{D(u, u, v)}$$

$$\leq \frac{\begin{array}{l} a_1\, D(u, u, v) + a_2\, \{D(u, u, v) + D(u, u, v)\} \\ + a_3\{D(u, v, v) + D(u, u, v)\} + a_4\{D(u, u, v) + D(u, u, v)\} \\ + a_5\{D(u, u, v)\, D(u, u, v) + D(u, u, v)\, D(u, u, v)\} \end{array}}{D(u, u, v)}$$

or, $(1 - 2a_2 - 2a_3 - 2a_4 - 2a_5)\, D(u, u, v) \leq 0$.

This implies that $u = v$. Therefore S_n, T_n and P_n have unique common fixed point in X.

Corollary 1. Let (X, D) be a complete and bounded D-metric space, let $S, T, P : X \to CB(X)$ be a multi-valued and orbitally continuous mapping such that S, T and P are pair-wise disjoint self-mappings of X, satisfying

(i) $$H(Sx, Ty, Pz) \leq a_1\, D(x, y, z) + a_2\{D(Sx, y, z) + D(x, Sx, z)\}$$
$$\frac{\begin{array}{l} + a_3\{D(y, z, Pz) + D(Sx, Ty, z)\} \\ + a_4\{D(Sx, y, Pz) + D(x, Ty, Pz)\} \\ + a_5\{D(x, Sx, z)\, D(y, Ty, z) + D(y, Sx, z)\, D(x, Ty, z)\} \end{array}}{D(x, y, z)}$$

for all distinct x, y, z in X and non-negative reals a_1, a_2, a_3, a_4 and a_5 such that

$(a_1 + 2a_2 + 2a_3 + 2a_4 + 2a_5) < 1$ with $(1/2) < (arp)^{J+1} < 1$

where $a = \dfrac{(a_1 + 2a_2 + a_4 + a_5)}{(1 - 2a_3 - 2a_4 - a_5)}$, $\qquad r = \dfrac{(a_1 + 2a_2 + 2a_3 + a_4 + a_5)}{(1 - a_3 - a_4 - a_5)}$

$$p = \frac{(a_1 + a_3 + a_4)}{(1 - a_2 - a_3 - 2a_4 - a_5)}$$

Then S, T and P have a unique common fixed point in X.

Proof : Proof follows immediately from Theorem 3.1.

REFERENCES

1. Bose, R. K. and Mukherjee, R. N. Common fixed point of some multi-valued mappings. Tamkang. *J.* Math., 81 (1977), 245-249.
2. Dhage, B. C., Generalized metric spaces and mappings with fixed points, Bull. Cal. Math. Soc. 84 (1992), 329-336.
3. Garegnani, G. and Massa, S. Multi-valued mappings of contractive type, Instit. Lombardo Accad. Sci. Rend., 112 (1978), 283-288.
4. Nadler, S. B.: Multi-valued contraction mappings, Pacific *J.* Math 30 (1969),475-488.
5. Veerapandi, T. and Rao Chandrasekhra, K. : Fixed point theorems of some multivalued mappings in D-metric space, Bull. Cal. Math. Soc. 87 (1995), 549- 556.

[1] Government P. G. College Sehore M.P.
[2] Department of mathematics, Unique college,
83, type-III, B-sector, Piplani, B.H.E.L. Bhopal M.P.
[3] Govt. Science And Commerce College, Benazir Bhopal M.P.

Jour. Pure Math., Vol. 21, 2004, pp. 101-110

ON WEAKLY PREOPEN FUNCTIONS IN BITOPOLOGICAL SPACES

TAKASHI NOIRI AND VALERIU POPA

Abstract : As a generalization of preopen functions, we introduce the notion of weakly preopen functions in bitopological spaces and obtain several characterizations and some properties of weakly preopen functions.

Key words and phrases : bitopological spaces, preopen sets, preopen function, weakly preopen function.

AMS(2000) Mathematics Subject Classification : 54C08, 54E55.

1. INTRODUCTION

The notion of preopen sets due to Mashhour et al. [12] play a significant role in general topology. In [3], [6] and [9] the concepts of preopen sets and preopen functions in bitopological spaces are introduced and investigated. In 1985, Rose [15] defined the notion of weakly open functions. Recently, Caldas and Navalagi [2] have introduced and studied the concept of weakly preopen functions between topological spaces.

In this paper we extend the results established by Caldas and Navalagi to functions between bitopological spaces. We obtain some characterizations and several properties of these functions.

2. PRELIMINARIES

Throughout the present paper, (X, τ_1, τ_2) (resp. (X, τ)) denote a bitopological (resp. topological) space. Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $Cl(A)$ and $\mathrm{Int}(A)$, respectively. Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. The closure of A and the interior of A with respect to τ_i are denoted by $iCl(A)$ and $i\mathrm{Int}(A)$, respectively, for $i = 1,2$.

Definition 2.1. A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) *(i, j)-regular open* [1] if $A = i\,\mathrm{Int}(j\,Cl(A))$, where $i \neq j$, $i, j = 1, 2$,

(2) *(i, j)-semi-open* [11] if $A \subset jCl(i\mathrm{Int}(A))$, where $i \neq j$, $i, j = 1, 2$,

(3) (i, j)-*preopen* [3] if $A \subset i\mathrm{Int}\,(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$,

(4) (i, j)-α-*open* [4] if $A \subset i\mathrm{Int}\,(j\mathrm{Cl}(A))$, where $i \neq j$, $i, j = 1, 2$.

The complement of an (i, j)-preopen set is said to be (i, j)-*preclosed* [6], [9]. A subset A is (i, j)-preclosed if and only if $i\mathrm{Cl}(j\mathrm{Int}(A)) \subset A$. The (i, j)-*preclosure* [9] of A, denoted by (i, j)-$p\mathrm{Cl}(A)$, is defined by the intersection of all (i, j)-preclosed sets containing A. The (i, j)-*preinterior* of A, denoted by (i, j)-$p\mathrm{Int}(A)$, is defined by the union of all (i, j)-preopen sets contained in A.

Lemma 2.1. *Let (X, τ_1, τ_2) be a bitopological space and $\{A_\lambda : \lambda \in \Lambda\}$ a family of subsets of X.*

(1) *If A_λ is (i, j)-preopen for each $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda$ is (i, j)-preopen,*

(2) *If A_λ is (i, j)-preclosed for each $\lambda \in \Lambda$, then $\cup_{\lambda\in\Lambda} A_\lambda$ is (i, j)-preclosed.*

Proof. (1) The proof follows from Theorem 4.2 of [6] and Theorem 3.2 of [9].

(2) This is an immediate consequence of (1).

Lemma 2.2. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.*

(1) (i, j)-$p\mathrm{Int}(A)$ *is (i, j)-preopen,*

(2) (i, j)-$p\mathrm{Cl}(A)$ *is (i, j)-preclosed,*

(3) *A is (i, j)-preopen if and only if $A = (i, j)$-$p\mathrm{Int}(A)$,*

(4) *A is (i, j)-preclosed if and only if $A = (i, j)$-$p\mathrm{Cl}(A)$.*

Proof. (1) and (2) follow from Lemma 2.1.

(3) and (4) follow from (1) and (2).

Lemma 2.3. *Let (X, τ_1, τ_2) be a bitopological space and A a subset of X.*

(1) $X - (i, j)$-$p\mathrm{Int}(A) = (i, j)$-$p\mathrm{Cl}(X - A)$,

(2) $X - (i, j)$-$p\mathrm{Cl}(A) = (i, j)$-$p\mathrm{Int}(X - A)$.

Proof. (1) By Lemma 2.2, (i, j)-$p\mathrm{Cl}(A)$ is (i, j)-preclosed. Then $X - (i, j)$-$p\mathrm{Cl}(A)$ is (i, j)-preopen. On the other hand, $X - (i, j)$-$p\mathrm{Cl}(X - A) \subset A$ and hence $X - (i, j)$-$p\mathrm{Cl}(X - A) \subset (i, j)$-$p\mathrm{Int}(A)$. Conversely, Let $x \in (i, j)$-$p\mathrm{Int}(A)$. Then there exists an (i, j)-preopen set G such that $x \in G \subset A$. Then $X - G$ is (i, j)-preclosed and $X - A \subset X - G$. Since $x \notin X - G$, $x \notin (i, j)$-$p\mathrm{Cl}(X - A)$ and hence (i, j)-$p\mathrm{Int}(A) \subset X - (i, j)$-$p\mathrm{Cl}(X - A)$. Therefore, $X - (i, j)$-$p\mathrm{Int}(A) = (i, j)$-$p\mathrm{Cl}(X - A)$. (2) This follows from (1) immediately.

Definition 2.2 Let (X, τ_1, τ_2) be a bitopological space and A a subset of X. A point x of X is said to be in the (i, j)-θ-*closure* [7] of A, denoted by (i, j)-$\mathrm{Cl}_\theta(A)$, if $A \cap j\mathrm{Cl}(U) \neq \phi$ for

every τ_i-open set U containing x, where $i, j = 1, 2$ and $i \neq j$.

A subset A of X is said to be (i, j)-θ-*closed* if $A = (i, j)\text{-}Cl_\theta(A)$. A subset A of X is said to be (i, j)-θ-*open* if $X - A$ is (i, j)-θ-closed. The (i, j)-θ-*interior* of A, denoted by $(i, j)\text{-Int}_\theta(A)$, is defined as the union of all (i, j)-θ-*open* sets contained in A. Hence $x \in (i, j)\text{-Int}_\theta(A)$ if and only if there exists a τ_i-open set U containing x such that $x \in U \subset jCl(U) \subset A$.

Lemma 2.4 *For a subset A of a bitopological space* (X, τ_1, τ_2), *the following properties hold:*

(1) $X - (i, j)\text{-Int}_\theta(A) = (i, j)\text{-}Cl_\theta(X - A)$,

(2) $X - (i, j)\text{-}Cl_\theta(A) = (i, j)\text{-Int}_\theta(X - A)$.

Lemma 2.5 (Kariofillis [7]). *Let* (X, τ_1, τ_2) *be a bitopological space. If U is a* τ_j*-open set of X, then* $(i, j)\text{-}Cl_\theta(U) = iCl(U)$.

Definition 2.3 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be

(1) (i, j)-*semi-open* [4] if for each τ_i-open set U of X, $f(U)$ is (i, j)-semi-open in Y,

(2) (i, j)-*preopen* [6]) if for each τ_i-open set U of X, $f(U)$ is (i, j)-preopen in Y,

(3) *weakly* (i, j)-*open* [5] if for each τ_i-open set U of X, $f(U) \subset i\,\text{Int}(f(jCl(U)))$.

Definition 2.4 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *weakly* (i, j)-*preopen* if for each τ_i-open set U of X, $f(U) \subset (i, j)\text{-}p\text{Int}\,(f(jCl(U)))$.

A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *pairwise weakly open* (resp. *pairwise weakly preopen*) if f is weakly (1, 2)-open and weakly (2, 1)-open (resp. weakly (1, 2)-preopen and weakly (2, 1)-preopen).

Remark 2.1 Since every τ_i-open set is (i, j)-preopen [9], every weakly (i, j)-open function is weakly (i, j)-preopen for $i, j = 1,2$ and $i \neq j$. The converse is not true as shown in Example 2.2 of [2].

3. CHARACTERIZATIONS

Theorem 3.1 *For a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *the following properties are equivalent :*

(1) *f is weakly (i, j)-preopen;*

(2) $f((i, j)\text{-Int}_\theta(A)) \subset (i, j)\text{-}p\text{Int}(f(A))$ *for every subset A of X;*

(3) $(i, j)\text{-Int}_\theta(f^{-1}(B)) \subset f^{-1}(i, j)\text{-}p\text{Int}(B))$ *for every subset B of Y;*

(4) $f^{-1}((i, j)\text{-}pCl(B)) \subset (i, j)\text{-}Cl_\theta(f^{-1}(B))$ *for every subset B of Y;*

(5) *For each $x \in X$ and each τ_i-open set U of X containing x, there exists an (i, j)-preopen set V containing $f(x)$ such that $V \subset f(j\mathrm{Cl}(U))$.*

Proof. (1) $\Rightarrow$ (2) : Let A be any subset of X and $x \in (i, j)\text{-}\mathrm{Int}_\theta(A)$. Then there exists a τ_i-open set U of X such that $x \in U \subset j\mathrm{Cl}(U) \subset A$. Hence we have $f(x) \in f(U) \subset f(j\mathrm{Cl}(U)) \subset f(A)$. Since f is weakly (i, j)-preopen, $f(U) \subset (i, j)\text{-pInt}\, f(j\mathrm{Cl}(U))) \subset (i, j)\text{-pInt}(f(A))$ and $x \in f^{-1}((i, j)\text{-pInt}(f(A)))$. Thus $(i, j)\text{-}\mathrm{Int}_\theta(A) \subset f^{-1}((i, j)\text{-pInt}(f(A)))$ which implies $f((i, j)\text{-}\mathrm{Int}_\theta(A) \subset (i, j)\text{-pInt}(f(A))$.

(2) $\Rightarrow$ (3) : Let B be any subset of Y. Then by (2), we have $f((i, j)\text{-}\mathrm{Int}_\theta(f^{-1}(B)) \subset (i, j)\text{-pInt}(f(f^{-1}(B))) \subset (i, j)\text{-pInt}(B)$. Therefore, $(i, j)\text{-}\mathrm{Int}_\theta(f^{-1}(B))) \subset f^{-1}((i, j)\text{-pInt}(B))$.

(3) $\Rightarrow$ (4) : Let B be any subset of Y. Then by $X - (i, j)\text{-}\mathrm{Cl}_\theta(f^{-1}(B)) = (i, j)\text{-}\mathrm{Int}_\theta(X - f^{-1}(B)) = (i, j)\text{-}\mathrm{Int}_\theta(f^{-1}(Y - B)) \subset f^{-1}(i, j)\text{-pInt}(Y - B))) = f^{-1}(Y - (i, j)\text{-p}\mathrm{Cl}(B)) = X - f^{-1}(i, j)\mathrm{pCl}(B))$. Therefore, $f^{-1}(i, j)\text{-p}\mathrm{Cl}(B)) \subset (i, j)\text{-}\mathrm{Cl}_\theta\ (f^{-1}(B))$.

(4) $\Rightarrow$ (5) : Let $x \in X$ and U be any τ_i-open set containing x. Set $B = Y - f(j\mathrm{Cl}(U))$. By (4), we have $f^{-1}((i, j)\text{-p}\mathrm{Cl}(Y - f(j\mathrm{Cl}(U)))) \subset (i, j)\text{-}\mathrm{Cl}_\theta(f^{-1}(Y - f(j\mathrm{Cl}(U)))$. Now, $f^{-1}((i, j)\text{-p}\mathrm{Cl}(Y - f(j\mathrm{Cl}(U))) = X - f^{-1}((i, j)\text{-pInt}(f(j\mathrm{Cl}(U))))$. Moreover, by Lemma 2.6 we have

$(i, j)\text{-}\mathrm{Cl}_\theta(f\text{-}1(Y - f(j\mathrm{Cl}(U)))) = (i, j)\text{-}\mathrm{Cl}_\theta(X - f^{-1}(f(j\mathrm{Cl}(U)))) \subset (i, j)\text{-}\mathrm{Cl}_\theta(X - j\mathrm{Cl}(U)) = i\mathrm{Cl}(X - j\mathrm{Cl}(U)) = X - i\mathrm{Int}(j\mathrm{Cl}(U)) \subset X - i\mathrm{Int}(U) = X - U$.

Therefore, we obtain $U \subset f^{-1}((i, j)\text{-pInt}(f(j\mathrm{Cl}(U))))$ and $f(U) \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(U))$. Since $f(x) \in f(U)$, there exists an (i, j)-preopen set V such that $f(x) \in V \subset f(j\mathrm{Cl}(U))$.

(5) $\Rightarrow$ (1) : Let U be any τ_i-open set of X and $x \in U$. By (5), there exists an (i, j)-preopen set V containing $f(x)$ such that $V \subset f(j\mathrm{Cl}(U))$. Hence we have $f(x) \in V \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(U)))$ for each $x \in U$. Therefore, we obtain $f(U) \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(U)))$. This shows that f is weakly (i, j)-preopen.

Theorem 3.2 *For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent :*

(1) *f is weakly (i, j)-preopen;*

(2) *$f(i\mathrm{Int}(F)) \subset (i, j)\text{-pInt}(f(F))$ for each τ_j-closed set F of X;*

(3) *$f(U) \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(U)))$ for every (i, j)-preopen set U of X;*

(4) *$f(U) \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(U)))$ for every (i, j)-α-open set U of X.*

Proof. (1) $\Rightarrow$ (2) : Assume that f is weakly (i, j)-preopen. Let F be a τ_j-closed set of X. Then $i\mathrm{Int}(F)$ is τ_i-open and by (1) we have $f(i\mathrm{Int}(F)) \subset (i, j)\text{-pInt}(f(j\mathrm{Cl}(F))) = (i, j)\text{-pInt}(f(F))$.

(2) $\Rightarrow$ (3) : Let U be any (i, j)-preopen set of X. Then by (2) we obtain $f(U) \subset j(i\,\mathrm{Int}(j\mathrm{Cl}(U)) \subset (i, j)\text{-pInt}\ (f\,(j\mathrm{Cl}(U)))$.

(3) $\Rightarrow$ (4) : This is obvious since every (i, j)-α-open set is (i, j)-preopen.

(4) $\Rightarrow$ (1) : Let U be any τ_i-open set of X. Then U is (i, j)-α-open in X and hence $f(U) \subset (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Therefore, f is weakly (i, j)-preopen.

Remark 3.1 Let $\tau = \tau_1 = \tau_2$ and $\sigma = \sigma_1 = \sigma_2$. Then Theorems 2.3, 2.4 and 2.5 of [2] for a function $f : (X,\tau) \to (Y, \sigma)$ follow from Theorems 3.1 and 3.2.

Theorem 3.3 *For a bijective function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent :*

(1) *f is weakly (i, j)-preopen;*

(2) *$(i, j)\text{-}p\mathrm{Cl}(f(j\mathrm{Int}(F))) \subset f(F)$ for every τ_i-closed set F of X;*

(3) *$(i, j)\text{-}p\mathrm{Cl}(f(U)) \subset f(i\mathrm{Cl}(U))$ for every τ_j-open set U of X.*

Proof. (1) $\Rightarrow$ (2) : Let F be any τ_i-closed set of X. Then $X - F$ is τ_i-open and

$$Y - f(F) = f(X - F) \subset (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(X - F))) =$$

$(i, j)\text{-}p\mathrm{Int}(f(X - j\mathrm{Int}(F))) = (i, j)\text{-}p\mathrm{Int}(Y - f(j\mathrm{Int}(F)) = Y - (i, j)\text{-}p\mathrm{Cl}(f(j\mathrm{Int}(F)).$

This implies that $(i, j)\text{-}p\mathrm{Cl}(f(j\mathrm{Int}(F))) \subset f(F)$.

(2) $\Rightarrow$ (3) : Let U be any τ_j-Open set of X. By (2) we have

$(i, j)\text{-}p\mathrm{Cl}(f(U)) = (i, j)\text{-}p\mathrm{Cl}(f(j\mathrm{Int}(U))) \subset (i, j)\text{-}p\mathrm{Cl}(f(j\mathrm{Int}(i\mathrm{Cl}(U)))) \subset f(i\mathrm{Cl}(U)).$

Therefore, $(i, j)\text{-}p\mathrm{Cl}(f(U)) \subset f(i\mathrm{Cl}(U))$.

(3) $\Rightarrow$ (1) : Let U be any τ_j-Open set of X. Then, we have

$Y - (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(U))) = (i, j)\text{-}p\mathrm{Cl}(Y - f(j\mathrm{Cl}(U))) = (i, j)\text{-}p\mathrm{Cl}(f(X - j\mathrm{Cl}(U))) \subset$

$f(i\mathrm{Cl}(X - j\mathrm{Cl}(U))) = f(X - i\mathrm{Int}(j\mathrm{Cl}(U))) \subset f(X - i\mathrm{Int}(U)) = f(X - U) = Y - f(U).$

This implies $f(U) \subset (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(U))$. Therefore, f is weakly (i, j)-preopen.

4. RELATIONS WITH OTHER FORMS OF OPENNESS

Definition 4.1 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *almost (i, j)-preopen* if $f(U)$ is (i, j)-preopen in (Y, σ_1, σ_2) for every (i, j)-regular open set U of (X, τ_1, τ_2).

Theorem 4.1 *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-preopen, then f is almost (i, j)-preopen.*

Proof. Let U be any (i, j)-regular open set of X. Then $U = i\mathrm{Int}(j\mathrm{Cl}(U))$ and hence U is τ_i-open. Therefore, $f(U)$ is (i, j)-preopen. This shows that f is almost (i, j)-preopen.

Theorem 4.2 *If a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is almost* (i, j)*-preopen, then* f *is weakly* (i, j)*-preopen.*

Proof. Suppose that f is almost (i, j)-preopen. For every τ_i-open set U of X, since $i\mathrm{Int}(jC1(U))$ is (i, j)-regular open, $f(i\mathrm{Int}(jC1(U)))$ is (i, j)-preopen in Y. Moreover, we have $f(U) \subset f(i\mathrm{Int}(jC1(U))) \subset f(jC1(U))$ and hence $f(U) \subset (i, j)\text{-}p\mathrm{Int}\,(f(jC1(U)))$. Therefore, f is weakly (i, j)-preopen.

Remark 4.1 a) The converses of Theorems 4.1 and 4.2 are not true in general. We can see the reason in Examples 2.2 and 2.19 of [2].

b) Let $\tau_1 = \tau_2 = \tau$ and $\sigma_1 = \sigma_2 = \sigma$. Then, by Theorem 4.2 we obtain Theorem 2.18 of [2] concerning a function $f : (X, \tau) \to (Y, \sigma)$.

Definition 4.2 A bitopological space (X, τ_1, τ_2) is said to be (i, j)*-almost regular* [16] if for each $x \in X$ and each (i, j)-regular open set U oontaining x, there exists an (i, j)-regular open set V of X: such that $x \in V \subset jC1(V) \subset U$.

Theorem 4.3 : *Let a bitopological space* (X, τ_1, τ_2) *be* (i, j)*-almost regular. Then a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is almost* (i, j)*-preopen if and only if it is weakly* (i, j)*-preopen.*

Proof. *Necessity.* This is shown by Theorem 4.2.

Sufficiency. Suppose that f is weakly (i, j)-preopen. Let U be any (i, j)-regular open set of X. Since X is (i, j)-almost regular, for eaeh $x \in U$ there exists an (i, j)-regular open set U_x such that $x \in U_x \subset jC1(U_x) \subset U$. Since every (i, j)-regular open set is τ_i-open and f is weakly (i, j)-preopen, we obtain

$$\begin{aligned} f(U) &= \cup\{f(U_x) : x \in U\} \subset \cup\{(i, j)\text{-}p\mathrm{Int}(f(j\,C1(U_x))) : x \in U\} \\ &\subset (i, j)\text{-}p\mathrm{Int}(\cup\{f(jC1(U_x)) : x \in U\}) = (i, j)\text{-}p\mathrm{Int}(f(\cup\{jC1(U_x) : x \in U\})) \\ &\subset (i, j)\text{-}p\mathrm{Int}(f(U)). \end{aligned}$$

By Lemma 2.2, $f(U)$ is (i, j)-preopen. Therefore, f is almost (i, j)-preopen.

Definition 4.3 A bitopological space (X, τ_1, τ_2) is said to be (i, j)*-regular* [8] if for each $x \in X$ and each τ_i-open set U containing x, there exists a τ_i-open set V such that $x \in V \subset jC1(V) \subset U$.

Theorem 4.4 *Let a bitopological space* (X, τ_1, τ_2) *be an* (i, j)*-regular. Then a function* $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ *is* (i, j)*-preopen if and only if it is weakly* (i, j)*-preopen.*

Proof. The proof is similar to that of Theorem 4.3.

Corollary 4.1 *Let (X, τ_1, τ_2) be an (i,j)-regular space. For a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent :*

(1) *f is (i, j)-preopen;*

(2) *f is almost (i, j)-preopen;*

(3) *f is weakly (i, j)-preopen.*

Remark 4.2 By Theorem 4.4, we obtain Theorem 2.12 of [2] which is a property of a function $f : (X, \tau) \to (Y, \sigma)$.

Definition 4.4 A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *strongly continuous* [10] if $f(Cl(A)) \subset f(A)$ for every subset A of X.

Theorem 4.5 *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (i, j)-preopen and strongly j-continuous, then f is (i, j)-preopen.*

Proof. Let U be any τ_i-open set of X. Since f is weakly (i, j)-preopen and strongly j-continuous, we have $f(U) \subset (i, j)\text{-}p\mathrm{Int}(f(jCl(U))) \subset (i, j)\text{-}p\mathrm{Int}(f(U))$. Therefore, $f(U) = (i, j)\text{-}p\mathrm{Int}(f(U))$. By Lemma 2.2, $f(U)$ is (i, j)-preopen. Hence f is (i, j)-preopen.

Definition 4.5 A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to have *the weak (i, j)-pinteriority condition* if $(i, j)\text{-}p\mathrm{Int}(f(j\,Cl(U))) \subset f(U)$ for every τ_i-open set U of X.

Theorem 4.6 *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (i, j)-preopen and satisfies the weak (i, j)-pinteriority condition, then f is (i, j)-preopen.*

Proof. Let U be any τ_i-open set of X. Since f is weakly (i, j)-preopen and satisfies the weak (i, j)-pinteriority condition, we have $f(U) \subset (i, j)\text{-}p\mathrm{Int}\, f(jCl(U))) \subset f(U)$. Therefore, $f(U) = (i, j)\text{-}p\mathrm{Int}(f(jCl(U)))$. By Lemma 2.2, $f(U)$ is (i, j)-preopen. Hence f is (i, j)-preopen.

5. SOME PROPERTIES OF WEAKLY (i, j)-PREOPEN FUNCTIONS

Definition 5.1 A bitopological space (X, τ_1, τ_2) is said to be *(i, j)-hyperconnected* if $j\,Cl(U) = X$ for every τ_i-open set U of X.

Theorem 5.1 *Let (X, τ_1, τ_2) be an (i, j)-hyperconnected space. Then a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is weakly (i, j)-preopen if and only if $f(X)$ is (i, j)-preopen in Y.*

Proof. *Necessity.* Let f be weakly (i, j)-preopen. Since X is τ_i-open, $f(X) \subset (i, j)\text{-}p\mathrm{Int}(f(j\,Cl(X))) = (i, j)\text{-}p\mathrm{Int}(f(X))$ and hence $f(X) = (i, j)\text{-}p\mathrm{Int}(f(X))$. By Lemma 2.2, $f(X)$ is (i, j)-preopen in Y.

Sufficiency. Suppose that $f(X)$ is (i, j)-preopen in Y. Let U be τ_i-open in X. Then $f(U) \subset f(X) = (i, j)\text{-}p\mathrm{Int}(f(X)) = (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(U))$. Therefore, $f(U) \subset (i, j)\text{-}p\mathrm{Int}(f(j\,\mathrm{Cl}(U)))$. This shows that f is weakly (i, j)-preopen.

Remark 5.1 By Theorem 5.1, we obtain Theorem 2.25 of [2] which is a property of a function $f : (X, \tau) \to (Y, \sigma)$.

Definition 5.2 A function $f: (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is said to be *(i, j)-contra-closed if $f(F)$* is σ_i-open in Y for every τ_j-closed set F of X.

Theorem 5.2 *If a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is (i, j)-contra-closed, then f is weakly (i, j)-preopen.*

Proof. Let U be any T_i-open set of X. Then $j\mathrm{Cl}(U)$ is τ_j-closed in X. We have $f(j\mathrm{Cl}(U)) = i\,\mathrm{Int}(f(j\mathrm{Cl}(U))) \subset i\,\mathrm{Int}(j\,\mathrm{Cl}(f(j\,\mathrm{Cl}(U))))$. Therefore, $f(j\mathrm{Cl}(U))$ is (i, j)-preopen. Hence $f(U) \subset f(j\mathrm{Cl}(U)) = (i, j)\text{-}p\mathrm{Int}(f(j\,\mathrm{Cl}(U)))$. Therefore, f is weakly (i, j)-preopen.

Lemma 5.1 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a bijective and (i, j)-semi-open function, then $j\,\mathrm{Int}(i\,\mathrm{Cl}\,(F))) \subset f(F)$ for every τ_i-closed set F of X.*

Proof. Let F be any τ_i-closed set of X. Then $X - F$ is τ_i-open in X. Since f is (i, j)-semi-open, $f(X - F) \subset j\,\mathrm{Cl}(i\mathrm{Int}\,(f(X - F))$. Therefore, $Y - f(F) = f(X - F) \subset j\mathrm{Cl}(i\,\mathrm{Int}\,(f(X - F)) = Y - j\,\mathrm{Int}\,(i\,\mathrm{Cl}(f(F))$. Therefore, $j\,\mathrm{Int}(i\,\mathrm{Cl}(f(F)) \subset f(F)$.

Theorem 5.3 *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a weakly (i, j)-preopen and (j, i)-semiopen bijection, then f is weakly (i, j)-open.*

Proof. Let U be any τ_i-open set of X. Then $j\mathrm{Cl}(U)$ is τ_j-closed. By Lemma 5.1, $i\,\mathrm{Int}(j\,\mathrm{Cl}(f(j\,\mathrm{Cl}(U)) \subset f(j\,\mathrm{Cl}(U))$ and hence $i\,\mathrm{Int}(j\,\mathrm{Cl}(f(j\,\mathrm{Cl}(U))) \subset i\mathrm{Int}\,(f(j\,\mathrm{Cl}(U))$. Since f is weakly (i, j)-preopen, $f(U) \subset (i, j)\text{-}p\mathrm{Int}(f(j\,\mathrm{Cl}(U))))$. By Lemma 2.2, $(i, j)\text{-}p\mathrm{Int}\,(f(j\mathrm{Cl}(U))))$ is (i, j)-preopen and hence $(i, j)\text{-}p\mathrm{Int}(f(j\,\mathrm{Cl}(U))) \subset i\,\mathrm{Int}(j\mathrm{Cl}\,((i, j)\text{-}p\mathrm{Int}(f(j\,\mathrm{Cl}(U))))) \subset i\mathrm{Int}\,(j\mathrm{Cl}(f(j\mathrm{Cl}(U))))$. Therefore, $f(U) \subset i\,\mathrm{Int}(j\,\mathrm{Cl}(f(j\mathrm{Cl}(U))) \subset i\mathrm{Int}(f(j\mathrm{Cl}(U)))$. Hence f is weakly (i, j)-open.

Remark 5.2 Theorem 5.3 is a dual form of Theorem 6.3 of [13] for a bijection $f : (X, \tau) \to (Y, \sigma)$.

Definition 5.3 A bitopological space (X, τ_1, τ_2) is said to be *pairwise connected* [14] (resp. *pairwise preconnected*) if it cannot be expressed as the union of two nonempty disjoint sets U and V such that U is τ_i-open and V is τ_j-open (resp. U is (i, j)-preopen and V is (j, i)-preopen).

Theorem 5.4 *(Y, σ_1, σ_2) is pairwise preconnected and $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a pairwise weakly preopen bijection, then (X, τ_1, τ_2) is pairwise connected.*

Proof. Suppose that (X, τ_1, τ_2) is not pairwise connected. There exist a τ_i-open set U_1 and a τ_j-open set U_2 such that $U_1 \neq \phi$, $U_2 \neq \phi$, $U_1 \cap U_2 = \phi$ and $U_1 \cup U_2 = X$. Hence, we have $f(U_1) \neq \phi$, $f(U_2) \neq \phi$, $f(U_1) \cap f(U_2) = \phi$ and $f(U_1) \cup f(U_2) = Y$. Since f is pairwise weakly preopen, $f(U_1) \subset (i, j)\text{-}p\mathrm{Int}(f(j\mathrm{Cl}(U_1)))$ and $f(U_2) \subset (j, i)\text{-}p\mathrm{Int}(f(i\mathrm{Cl}(U_2)))$. Since U_1 and U_2 are τ_j-closed and τ_i-closed, respectively, we have $f(U_1) \subset (i, j)\text{-}p\mathrm{Int}(f(U_1))$ and $f(U_2) \subset (j, i)\text{-}p\mathrm{Int}(f(U_2))$ and hence $f(U_1) = (i, j)\text{-}p\mathrm{Int}(f(U_1))$ and $f(U_2) = (j, i)\text{-}p\mathrm{Int}(f(U_2))$. By Lemma 2.2, $f(U_1)$ is (i, j)-preopen and $f(U_2)$ is (j, i)-preopen. This is contrary that (Y, σ_1, σ_2) is pairwise preconnected. Therefore, (X, τ_1, τ_2) is pairwise connected.

Remark 5.3 By Theorem 5.4, we obtain Theorem 2.23 of [2] which is a property of a function $f : (X, \tau) \to (Y, \sigma)$.

REFERENCES

1. S. Bose and D. Sinha, *Almost open, almost closed, θ-continuous and almost compact mappings in bitopological spaces*, Bull. Calcutta Math. Soc. 73 (1981), 345-354.
2. M. Caldas and G. Navalagi, *On weakly preclosed functions*, Arch. Math. (Brno) 40 (2004), 119-128.
3. M. jelić, *A decomposition of pairwise continuity*, J. Inst. Math. Comput. Sci. Math. Ser. 3 (1990), 25-29.
4. M. Jelić, *Feebly p-continuous mappings*, Suppl. Rend. Circ. Mat. Palermo (2) 24 (1990), 387-395.
5. M. Jelić, *On some mappings of bitopological spaces*, Suppl. Rend. Circ. Mat. Palermo (2) **29** (1992), 483-494.
6. A. Kar and P. Bhattacharyya, *Bitopological preopen sets, precontinuity and preopen* mappings, Indian J. Math. 34 (1992), 295-309.
7. C. G. Kariofillis, *On pairwise almost compactness*, Ann. Soc. Sci. Bruxelles **100** (1986), 129-137.
8. J. C. Kelly, *Bitopological spaces*, Proc. London Math. Soc. (3) **13** (1963), 71-89.
9. F. H. Khedr, S. M. Al-Areefi and T. Noiri, *Precontinuity and semi-precontinuity in bitopalogical spaces*, Indian J. Pure Appl. Math. 23 (1992), 625-633.
10. N. Levine, *Strong continuity in topological spaces*, Amer. Math. Monthly **67** (1960), 269.
11. S. N. Maheshwari and R. Prasad, *Semi open sets and semi continuous functions in bitopological spaces*, Math. Notae **26** (1977/78), 29-37.
12. A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb, *On precontinuous and weak precontinuous mappings*, Proc. Math. Phys. Soc. Egypt. 53 (1982), 47-53.

13. C. Paul and P. Bhattacharyya, *Properties of quasi-precontinuous functions,* Indian J. Pure Appl. Math. **27** (1996), 475-486.

14. W. J. Pervin, *Connectedness in bitopological spaces,* Indag. Math. **29** (1967), 369-372.

15. D. A. Rose, *On weak openness and almost openness,* Internat. J. Math. Math. Sci. 7 (1984), 35-40.

16. A. R. Singal and S. P. Arya, *On pairwise almost regular spaces,* Glasnik Mat. Ser. III 6 (26) (1971), 335-343.

Takashi NOIRI
2979-1 Shiokita-cho, Hinagu,
Yatsushiro-shi, Kumamoto-ken,
869-5142 JAPAN
e-mail : t.noiri@nifty.com

Valeriu POPA
Department of Mathematics
Bacău University
5500 Bacău, RUMANIA
e-mail : vpopa@ub.ro

JOURNAL OF PURE MATHEMATICS

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF CALCUTTA

INSTRUCTIONS

- Manuscript (3 copies) submitted must have its title written on top followed by the name(s) of the author(s) in the next line, both in Roman Bold Capitals like

RINGS OF CONTINUOUS FUNCTIONS
L. GILLMAN AND M. JERISON

- This should be followed by a short abstract, not exceeding three hundred words, and then, by the latest AMS subject classification.
- Usage of footnotes should be avoided.
- In the main body of the manuscript references should be denoted by numbers enclosed within third brackets.
- The section headings should be in bold capitals and be centred, as below

6. THE MAIN THEOREM

- Theorem, lemma, corollary, proof etc. should be shown as **Theorem, Lemma, Corollary, Proof** etc.
- References should be listed at the end in alphabetical order of the surnames of the authors as follows.

REFERENCES

1. N. L. Alling, Foundations of Analysis on Surreal Number Fields, North-Holland Publishing Co., 1987.
2. E. Hewitt, Rings of continuous functions I, Trans. Amer. Math. Soc. 64(1948), 54-99.

- Address(es) of the author(s) should be given at the end.

JOURNAL OF PURE MATHEMATICS
UNIVERSITY OF CALCUTTA

VOLUME 21, 2004

CONTENTS

Price : 50/-

Published by the Registrar, University of Calcutta & Printed by Sri Pradip Kumar Ghosh, Superintendent, Calcutta University Press, 48, Hazra Road, Kolkata —700 019.